

NILS SVARTHOLM

THE BINDING ENERGIES OF THE
LIGHTEST ATOMIC NUCLEI

WITH AN APPLICATION OF THE THEORY OF INTEGRAL
EQUATIONS TO THE EIGENVALUE PROBLEMS

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INAUGURAL DISSERTATION

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BY

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Introduction.

A. Fundamental Properties of Nuclei.

The theories of the structure of atomic nuclei have been developed under circumstances which might be regarded as most favourable. On the one hand quantum theory has attained a state of maturity with respect to its understanding as well as to its definite success in applications. On the other hand experimental technique has furnished us with numerous facts concerning nuclei and nuclear processes. Notwithstanding this state of things it is an indisputable fact that our present knowledge of nuclear structure is very far from being a well-founded and consistent theory. One has the impression that definite progress in this field must be intimately associated with the resolution of some of the most serious fundamental difficulties of physics, above all those concerning the unification of quantum theory and relativity. Without attempting to analyse the development of nuclear theory we may endeavour, by a brief discussion, to make the present perplexing situation in some degree intelligible.

Most properties of nuclei have little direct bearing upon the properties of matter and radiation as manifested in other branches of physics and in daily life. Thus very little is known with certainty about the nature of nuclear forces, that is forces between nuclei and nuclear constituents, while the Coulomb law, indispensable to the theory of atomic structure, was known from the very dawn of electrical knowledge. With respect to atomic structure another reason may be presented, which concerns the accuracy of the empirical results. The accuracy of the experimental data in nuclear physics is as a rule very high in itself but exceedingly

poor in comparison with the capacities of ordinary spectroscopy. Furthermore, the data regarding a given nucleus and the attributes of that nucleus are rather small in number. For example, we know rather little about the energy levels of an excited nucleus. Among the empirical data concerning the energy levels of nuclei none show correspondence to a Balmer or a Paschen series. Frequently an empirical rule has lost its significance on the appearance of new or more accurate experimental data or interpretations.

Some of the most fundamental properties of nuclei are comprised by the following statements regarding nuclear constitution.

I) Any nucleus consists of a definite number (Z) of protons and a number (N) of neutrons. Protons and neutrons can be considered either as two distinct types of heavy elementary particles or as two states of one and the same particle, the so-called *nucleon*.

II) The total number of nucleons is equal to the *mass number* $A = Z + N$, *i.e.* the integer nearest the atomic weight of the nucleus, while the number of protons is identical with the nuclear charge number.

III) The *parity* of a nucleus, that is the parity of the mass number A , is a most characteristic property. Odd nuclei obey Fermi-Dirac statistics and have spin quantum numbers which are half integers, while even nuclei obey Bose-Einstein statistics and have integral spin quantum numbers.

IV) The atomic weight of a nucleus is smaller than the sum of the weights of its constituent nucleons by an amount which can be interpreted by Einstein's law of equivalence of mass and energy as the *binding energy* of the nucleus.

A considerable number of facts are explained by these statements. Their importance and significance can hardly be understood without making a close study of the historical development.

Other properties of nuclei such as the magnitude of the spin and its relation to the energy state, electrical and magnetic moments, interaction with radiation, stability against α - and β -disintegration and fission compel a much closer study especially of those properties of the nuclear constituents which we call forces. Nevertheless, a most thorough systematization of nuclear energy levels may be based upon the points mentioned above combined with some very general assumptions concerning the nuclear forces

as has been shown by HUND (1937) and WIGNER (1937). Their theory deals with the symmetry properties of wave functions describing a definite nuclear system and exhibits results irrespective of the true positions of the levels. This situation is very similar to that found in ordinary spectroscopy. The theory is, however, restricted in both cases by obvious limitations, and under any circumstances we must consider the nuclear force problem as being one of the most central in nuclear theory. This problem has been attacked essentially from two different quarters. On the one hand the simplest possible assumptions about the forces are made which can account for as many facts as possible, such as binding energies, stability rules and data from scattering experiments. On the other hand it has been attempted to derive the nuclear forces from a field theory in more or less close analogy to the derivation of the Coulomb law from Maxwell's theory of electromagnetism. The application of such forces in nuclear problems is, however, in some degree prohibited by the great complexity as well as the incompleteness of the theory. These two views, however, are by no means mutually exclusive. The suggestive ideas in the field theories can often be loosely reconciled with the more phenomenological points of view.

B. Nuclear Forces.

In the beginning some qualitative conclusions about the nuclear forces may be drawn from the stability rules. A necessary condition for the stability of a nucleus is that it must not be heavier than its parts after an arbitrary division. This rule clearly agrees with the statements in A.

A second rule (due to Mattauch) refers to nuclei with a definite mass number A , *i. e.* *isobars*, and states that only the lighter of two neighbouring isobars is stable. The instability of the heavier isobar presents itself not as a disintegration of the nucleus into other nuclei but as an emission of a β -particle transferring the nucleus into its isobaric neighbour. By employing the term »emission» instead of disintegration we have endeavoured to emphasize the generally accepted view according to which electrons do not exist within a nucleus but are formed at the very

moment of emission in analogy to the creation of photons at the emission of light from atoms. By this mode of expression the statements I—IV in A) are not invalidated. Now the isobar rule plays an important rôle in the force problem. Formerly (BETHE-BACHER 1936, § 6), the fact that all known nuclei have equal or nearly equal numbers of protons and neutrons was taken as an argument for the view that the force between a proton and a neutron be essentially larger than that between two protons or two neutrons. Other arguments speak strongly in favour of exact or almost exact equality of $p-p$, $p-n$ and $n-n$ forces. As a matter of fact, the only necessary conclusion which may be drawn from the empirical rule $Z \sim N$ is that the $p-p$ force is nearly equal to the $n-n$ force. A real difference between the $p-p$ force and the $n-n$ force is apparently the Coulomb repulsion force between the protons in the former case. This difference is quite sufficient to account for the excess of neutrons in heavier nuclei. On the other hand, the approximate equality of $p-n$, $p-p$ and $n-n$ forces may explain the empirical fact that the differences between the binding energies of isobaric nuclei are rather small. For two neighbouring isobars the energy difference is expressed by the maximum energy of the β -particle emitted by the heavier nucleus according to the isobar rule. By the latter rule the number of stable isobars is reduced to the actual ones in the vicinity of $Z=N$. Furthermore, it has been proved (BETHE 1938, ELLIOT and KING 1941) that the instability of nuclei with $Z-N=1$ is due solely to the differences between the Coulomb energies of the nuclei $Z-N=1$ and $Z-N=0$. The β -instability of other nuclei might possibly originate in other energy differences due to electrical or magnetic effects.

The equal-force hypothesis is strongly supported by scattering experiments. In themselves such experiments reveal some of the most characteristic properties of nuclear forces, above all their short range. Experiments on scattering of protons and neutrons in hydrogen have shown that there exists a strong attractive force between two protons (disregarding the repulsive Coulomb force) as well as between a proton and neutron within a range of about 10^{-13} cm and that these forces are equal. A possible difference cannot exceed a few per cent (BREIT, THAXTON and EISENBUD 1939). More accurate experimental data for a broader energy

range combined with a close analysis of the results would facilitate an exact determination of the forces in this way.

For the present we presumably gain more information about nuclear forces from binding energies. As a function of the mass number A , the binding energy per particle, that is nucleon, shows a rapid increase for the lightest nuclei H^2 , H^3 , He^3 and He^4 , from 1.09 Mev for H^2 to about 7 Mev for He^4 . The binding energy per particle thereafter remains roughly constant between the limits 7 and 8.5 Mev with fluctuations showing marked maxima at $A=4m$, where m is an integer. The constancy is the most outstanding feature, since ordinary forces such as electrical or magnetic ones would necessitate an approximately linear dependence between particle number and binding energy per particle. This so-called saturation property has given rise to a comparison of nuclear forces with chemical forces. Special interest has been devoted to the α -particle model according to which a nucleus is built up of a number of α -particles and some extra neutrons or protons. The principal part of the binding energy is contained in the α -particles, while the remainder of the binding energy and the stability of the whole nucleus is derived from forces of, for example, the van der Waals type. The chemical analogy cannot, however, be extended too far. The van der Waals attractive forces between molecules, in themselves of short range, are balanced by strong repulsive forces of still shorter range. Such forces probably exist between α -particles but likely not between nucleons. The scattering experiments mentioned above do not allow such an interpretation. Another factor borrowed from the chemical valence theory, the *exchange force*, has played an important rôle ever since the foundation of the theories of nuclear constitution by HEISENBERG in 1932. Nowadays, however, there seems to be experimental evidence against this hypothesis. As has been pointed out by WICK (1933) (see BB, p. 121) high-energy neutrons should preferentially be scattered backwards by protons if the $p-n$ force were of the exchange type. Available experimental results (AMALDI, BOCCIARELLI, FERRETTI and TRABACCHI 1942, CHAMPION and POWELL 1944) tend in an opposite direction but their evidence is possibly not convincing.¹ However the saturation property of nuclear forces

¹ I wish to thank Dr. S. Rozenthal for information on the latter point.

may be explained, since it does not directly concern the lightest elements, the hydrogen and helium isotopes, its discussion can be deferred until a better knowledge of the nuclear forces is attained by consideration of the properties of the latter nuclei. There is a further point which concentrates our attention on nuclei not heavier than the α -particle, namely the classification theory mentioned above.

According to the Pauli principle, valid for the nucleons, not more than two neutrons or two protons can be in the same orbital quantum state. Thus the α -particle with all its nucleons in an orbital ground state represents a closed configuration with high stability in analogy to the closed electron groups of the rare gas atoms. Such closed groups also show a strong mutual repulsion at small distances, which is in accordance with the α -particle model.

It then remains to find an explanation for the strong increase of binding energies in the series H^2 to He^4 . The view generally accepted (BB, § 9) is that the attractive force between two nucleons increases very rapidly with decreasing distance. In the classical conception one would say that the mean kinetic energy as well as the mean potential energy would be numerically quite large in comparison with the total energy if there were only two particles present while the potential energy would increase more rapidly than the kinetic energy when more particles were present. A quantitative test for this view requires the solution of the wave equations for the two-, three- and four-body problems with a potential energy operator constructed from known and assumed properties of the forces. Such simple forms as $e^{-r/a}$ or e^{-r^2/a^2} for the potential energy of two nucleons separated by the distance r are quite sufficient to start with. They have the characteristic range property quantitatively expressed by the length a and vanish, practically speaking, not far beyond this range. Though the methods available for solving the wave equations for the three- and four-body problems are slowly convergent approximate methods, the results indicate that the potential energy cannot be expressed by interaction terms so simple as those mentioned above. As these results are largely independent of the exact analytical form of the interaction terms, modifications have been attempted in other directions, especially in coupling the forces to the spins of the nucleons. These possibilities will be discussed in the present paper

in connection with the presentation of a new method of solving wave equations. We conclude this introduction with a short review of earlier methods and a discussion of their possibilities and their limitations.

C. Methods of Calculation.

The wave-mechanical n -body problem in nuclear physics differs from the corresponding problem in atomic physics especially in one respect: the short-range property of nuclear forces. Beyond the range of the forces, that is when the particles are sufficiently distant from each other in relation to the range, the wave functions have an exponential character, whereas they fluctuate heavily whenever two or more particles come closer together. As a matter of fact, analytical expressions with such peculiar qualities are very difficult to construct. The approximate method most frequently used in these problems, the variational method, works with trial wave functions containing one or more parameters, which are varied until a lowest possible energy is obtained. Sometimes linear function systems, more or less complete, are used. The total energy is then expressed as a quadratic form in the unknown expansion coefficients and the energy is minimized with respect to these coefficients. Frequently both methods are combined, so that a single function of the function system containing a parameter to be varied is responsible for the major part of the energy. By the inclusion of more members of the function system the energy can be lowered further. Bearing in mind the above remarks about the character of the wave functions, we cannot expect these methods to give results converging with any noteworthy rapidity. Actually the true lowest energy level is known in some cases, and suspected in others, to lie considerably below the computed energy. An advantage, characterizing the variational method, is that energies obtained with any wave functions are never below the correct energy level. Another method most useful in other respects, namely the equivalent two-body method of Feenberg (BB § 21), has not this advantage. In all cases an estimation of the error of an energy value obtained by means of an approximate solution provides a very difficult problem.

The Schrödinger perturbation theory has also been applied to the nuclear eigenvalue problems, above all for nuclei containing more than four particles. For the lightest nuclei the perturbation method has been used occasionally for comparison with results obtained by means of variational methods. The disadvantages of the perturbation method in nuclear problems are the following: 1) the first-order perturbation is a poor approximation, 2) the second-order perturbation (as well as all those of even order) does not give an upper bound of the energy, 3) the third-order perturbation is as a rule difficult to calculate.

Another manner of proceeding, not so often employed in quantum-mechanical problems, consists in a transformation of the eigenvalue problem to momentum space. This procedure applied to the nuclear force problems offers special advantages which have not been observed or utilized hitherto. (Cf. THOMAS 1937).

The wave functions of a nuclear eigenvalue problem are quite different in coordinate space and in momentum space. The outer region of coordinate space characterized by wave functions of an exponential type, determined only by the binding energy, corresponds in momentum space to an inner region where the wave functions decrease smoothly from a maximum at the origin. The narrow inner region of coordinate space with rapidly fluctuating wave functions corresponds to a broad outer region in momentum space. These circumstances will render the construction of approximate wave functions with simple analytical expressions in momentum space easier than in coordinate space. Furthermore, a special method of calculation can be applied to the nuclear eigenvalue problem in momentum space. This method is the main theme in the subsequent investigation (Chapters I and III).

A criterion for the usefulness of an approximation method is given by the two questions: 1) how great accuracy can be attained? 2) how many problems can be mastered? Up to now, in nuclear theory, one has enjoyed comparatively great freedom as to the choice of the interaction potential. Thus the error function potential e^{-r^2/a^2} , which has pronounced range properties, has been chosen frequently mainly because it makes an approximate solution possible for two, three as well as four particles. With other potentials one cannot go beyond three particles and even the three-particle problem is associated with great difficulties

in many cases. Now the nuclear force problem requires the solution of as many problems as possible with the simplest possible assumptions with regard to the potentials. We therefore might be justified in examining anew the two-, three- and four-particle problems with the error function potential with special reference to the accuracy (Chapters II and IV). Among earlier papers treating this subject we may mention especially those of FEENBERG (1935) and DOLCH (1936). (*Cf.* BB § 20.) The nuclear theory has, however, undergone some modifications since the appearance of those works. More up-to-date points of view are found in papers by FLÜGGE (1937) and by MARGENAU and collaborators (1937, 1938).

In Chapter V we discuss the corresponding problems for the so-called Yukawa potential $e^{-\kappa r}/\kappa r$. Though the accuracy is not so great in this case as in the case of the error function potential, definite conclusions can be drawn concerning the consistency of the results. In the last chapter the consequences of an extended force conception are discussed.

* *

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In publishing this thesis I take the opportunity of expressing my sincerest gratitude to all those who have assisted me in various ways during my work in the present as well as in earlier investigations.

My thanks are especially due to my teacher, Professor OSKAR KLEIN. He has never hesitated to place his knowledge at my disposal in private discussions and in colloquia. His great interest and profound experience in the fundamental problems of physics have been of the greatest importance in my studies.

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CHAPTER I.

The Nuclear Two-body Problem.

1. Formulation of the Problem in Momentum Space.

The Schrödinger equation of the deuteron is written in the following form

$$\left\{ \frac{p_1^2}{2M} + \frac{p_2^2}{2M} + V(r_{12}) \right\} \psi(\mathbf{r}_1 \mathbf{r}_2) = E \psi(\mathbf{r}_1 \mathbf{r}_2) \quad (1.1)$$

where $\mathbf{p}_1$ is the momentum of the proton and $\mathbf{p}_2$ the momentum of the neutron. We neglect the difference in the masses of the neutron and the proton and take, appropriately, M as the mean value of the two masses. For the sake of simplicity the potential energy $V(r_{12})$ is assumed to be a function only of the distance r_{12} between the proton and the neutron, though most of the considerations in this section are valid also in the case where V is a function of the vector $\mathbf{r}_{12}$. According to the fundamental conceptions of quantum mechanics the momentum $\mathbf{p}_1$ is interpreted as the differential operator $-i\hbar \text{grad}_1$ and p_1^2 accordingly as $-\hbar^2 \Delta_1$, referred to the space coordinates $\mathbf{r}_1 = (x_1 \ y_1 \ z_1)$ of the proton. The potential energy $V(r_{12})$ acts on $\psi(\mathbf{r}_1 \mathbf{r}_2)$ as an ordinary multiplier. When the whole problem (1.1) is transposed from the coordinate space $(\mathbf{r}_1 \mathbf{r}_2)$ to the momentum space $(\mathbf{p}_1 \mathbf{p}_2)$ by means of the transformation

$$\psi(\mathbf{r}_1 \mathbf{r}_2) \rightarrow \varphi(\mathbf{p}_1 \mathbf{p}_2) = \int \int (\mathbf{p}_1 | \mathbf{r}_1) (\mathbf{p}_2 | \mathbf{r}_2) \psi(\mathbf{r}_1 \mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2 \quad (1.2)$$

where

$$(\mathbf{p}_j | \mathbf{r}_j) = h^{-\frac{3}{2}} \exp \{ -i \mathbf{p}_j \cdot \mathbf{r}_j / \hbar \} \quad (j=1, 2) \quad (1.3)$$

we find that the squares of the momenta p_1^2 and p_2^2 act as ordinary multipliers while the potential energy $V(r_{12})$ acquires the more general form of an integral operator. For a strict mathe-

mathematical theory of this representation of the wave equation the reader is referred to a paper by GORDON (1934). If treated as a continuous matrix according to the transformation theory, $V(r_{12})$ has the following matrix elements

$$(\mathbf{p}_1 \mathbf{p}_2 | V | \mathbf{p}'_1 \mathbf{p}'_2) = \iint (\mathbf{p}_1 | \mathbf{r}_1) (\mathbf{p}_2 | \mathbf{r}_2) V(|\mathbf{r}_1 - \mathbf{r}_2|) (\mathbf{r}_1 | \mathbf{p}'_1) (\mathbf{r}_2 | \mathbf{p}'_2) d\mathbf{r}_1 d\mathbf{r}_2 \quad (1.4)$$

Thus the Schrödinger equation (1.1) has the following form in momentum space

$$\begin{aligned} & \frac{1}{2M} (p_1^2 + p_2^2) \varphi(\mathbf{p}_1 \mathbf{p}_2) + \\ & + \iint (\mathbf{p}_1 \mathbf{p}_2 | V | \mathbf{p}'_1 \mathbf{p}'_2) \varphi(\mathbf{p}'_1 \mathbf{p}'_2) d\mathbf{p}'_1 d\mathbf{p}'_2 = E \varphi(\mathbf{p}_1 \mathbf{p}_2) \end{aligned} \quad (1.5)$$

which is a linear and homogeneous integral equation.

A transformation of coordinates

$$\frac{\mathbf{r}_1 + \mathbf{r}_2}{2} = \mathbf{R} \quad \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}_{12} = \mathbf{r} \quad (1.6)$$

is applied to the integral of (1.4). With respect to the equation (1.3) and its conjugate

$$(\mathbf{r}_j | \mathbf{p}_j) = h^{-\frac{3}{2}} \exp\{i \mathbf{p}_j \cdot \mathbf{r}_j / \hbar\} \quad (j=1, 2) \quad (1.3')$$

we have

$$\begin{aligned} (\mathbf{p}_1 \mathbf{p}_2 | V | \mathbf{p}'_1 \mathbf{p}'_2) &= h^{-6} \iint \exp\{-i(\mathbf{p}_1 - \mathbf{p}'_1) \cdot \mathbf{r}_1 / \hbar - i(\mathbf{p}_2 - \mathbf{p}'_2) \cdot \mathbf{r}_2 / \hbar\} \cdot \\ &\cdot V(|\mathbf{r}_1 - \mathbf{r}_2|) d\mathbf{r}_1 d\mathbf{r}_2 = h^{-6} \int \exp\{-i(\mathbf{p}_1 - \mathbf{p}'_1 + \mathbf{p}_2 - \mathbf{p}'_2) \cdot \mathbf{R} / \hbar\} d\mathbf{R} \cdot \\ &\cdot \int \exp\{-i(\mathbf{p}_1 - \mathbf{p}'_1 - \mathbf{p}_2 + \mathbf{p}'_2) \cdot \mathbf{r} / 2\hbar\} V(|\mathbf{r}|) d\mathbf{r} \end{aligned} \quad (1.7)$$

The first factor is an improper integral, which can be expressed by the symbolic δ -function of Dirac according to

$$\begin{aligned} h^{-3} \int \exp\{-i \mathbf{P} \cdot \mathbf{R} / \hbar\} d\mathbf{R} &= \delta(\mathbf{P}) \\ \delta(\mathbf{P}) &= \delta(P_x) \delta(P_y) \delta(P_z) \end{aligned} \quad (1.8)$$

Thus we obtain

$$\begin{aligned} (\mathbf{p}_1 \mathbf{p}_2 | V | \mathbf{p}'_1 \mathbf{p}'_2) &= h^{-3} \delta(\mathbf{p}_1 - \mathbf{p}'_1 + \mathbf{p}_2 - \mathbf{p}'_2) \cdot \\ &\cdot \int \exp\{-i(\mathbf{p}_1 - \mathbf{p}'_1) \cdot \mathbf{r} / \hbar\} V(|\mathbf{r}|) d\mathbf{r} \end{aligned} \quad (1.9)$$

Now the integral equation (1.5) can be written

$$\frac{1}{2M}(p_1^2 + p_2^2)\varphi(\mathbf{p}_1\mathbf{p}_2) + h^{-3} \int \int \exp\{-i(\mathbf{p}_1 - \mathbf{p}'_1) \cdot \mathbf{r} / \hbar\} V(|\mathbf{r}|) d\mathbf{r} \cdot \\ \cdot \varphi(\mathbf{p}'_1, \mathbf{p}_1 - \mathbf{p}'_1 + \mathbf{p}_2) d\mathbf{p}'_1 = E\varphi(\mathbf{p}_1\mathbf{p}_2) \quad (1.10)$$

As we are interested only in the relative motion of the two particles in a deuteron, we can assume from the very beginning without loss of generality that the centre of gravity of the deuteron is at rest. The solution of the equation (1.10) then has the form

$$\varphi(\mathbf{p}_1\mathbf{p}_2) = \delta(\mathbf{p}_1 + \mathbf{p}_2) \varphi(\mathbf{p}_1) \quad (1.11)$$

If (1.11) is inserted in (1.10) the latter equation can be integrated with respect to $\mathbf{p}_2$

$$\frac{p_1^2}{M} \varphi(\mathbf{p}_1) + h^{-3} \int \int \exp\{-i(\mathbf{p}_1 - \mathbf{p}'_1) \cdot \mathbf{r} / \hbar\} V(|\mathbf{r}|) d\mathbf{r} \varphi(\mathbf{p}'_1) d\mathbf{p}'_1 = E\varphi(\mathbf{p}_1) \quad (1.12)$$

As may be expected, the solutions of this integral equation are quite different in the two cases $E > 0$ and $E < 0$. In the latter case, but not in the former, the equation (1.12) can be symmetrized. By putting

$$\varphi(\mathbf{p}) = (p^2 - ME)^{-\frac{1}{2}} \chi(\mathbf{p}) \quad (1.13)$$

and omitting the index 1, we obtain

$$\chi(\mathbf{p}) = \int S(\mathbf{p}|\mathbf{p}') \chi(\mathbf{p}') d\mathbf{p}' \quad (1.14)$$

which is a homogeneous integral equation of the second kind with the symmetrical kernel

$$S(\mathbf{p}|\mathbf{p}') = -Mh^{-3} (p^2 - ME)^{-\frac{1}{2}} (p'^2 - ME)^{-\frac{1}{2}} \cdot \\ \cdot \int \exp\{-i(\mathbf{p} - \mathbf{p}') \cdot \mathbf{r} / \hbar\} V(|\mathbf{r}|) d\mathbf{r} \quad (1.15)$$

For the potentials most frequently used in nuclear theory the integral equation (1.14) is only apparently singular. As a rule the infinite integration range can be transformed into a finite range with a resulting kernel finite in the whole range, except possibly for some unimportant singularities. Moreover, the kernels are positive-definite in most cases.

If the potential is introduced as a product

$$V(|\mathbf{r}|) = Af(r) \quad (1.16)$$

where A denotes the strength of the force, we can consider the constant A as an eigenvalue parameter of the integral equation (1.12) or (1.14), while the energy parameter E is assumed to be equal to the actual binding energy of the deuteron. According to the theory of integral equations the solutions of (1.14) form a discrete set with an appertaining set of eigenvalues $A_1, A_2, \dots$.

Evidently the ground state of the deuteron corresponds to the lowest eigenvalue of the set A . The remaining eigenvalues A correspond to excited states of a hypothetical configuration with a ground state lower than E . By this inversion of the whole eigenvalue problem we can treat the discrete ($E < 0$) and the continuous ($E > 0$) energy spectra separately. This fact is quite advantageous when variational methods are applied. Since an analytical solution of (1.14) is not possible in most cases, approximate methods must be used. Two methods of the integral equation theory are especially worthy of mention, namely the Gauss-Hilbert variational principle and the method of iterated functions due to Kellogg. By means of these methods sets of approximate solutions of (1.14) may be constructed.

The variational principle proves to be equivalent to the ordinary variational method of atomic and nuclear theory. This fact can easily be understood, since a lower bound of the energy for a fixed value of the force constant implies a lower bound of the latter if the energy is kept constant. The kinetic and potential energies are then, of course, assumed to be the same functions of the variational parameters in both cases. Actually, this is the case when the momentum wave function is the Fourier transform of the coordinate wave function. The wave functions obtained by the variational method (in momentum space) may then be used as initial functions in the method of iterated functions. The improvements in eigenfunctions and eigenvalues achieved by the latter method are thus always referred to results obtained by the variational method, whereby comparison with earlier results is rendered easier.

2. The Variation—Iteration Method.

The integral equation (1.12) is written in the form

$$T(p)\varphi(p) = \lambda \int K(p|p')\varphi(p')dp' \quad (2.1)$$

where the variable p represents the three-dimensional momentum space and $T(p)$ is a positive function. The kernel $K(p|p') = K^*(p'|p)$ is assumed to be positive-definite, which means that

$$\int \int f^*(p)K(p|p')f(p')dpdp' \geq 0 \quad (2.2)$$

for all functions $f(p)$.

By means of an arbitrary initial function $\varphi_0(p)$ a set of iterated functions $\varphi_1(p)$, $\varphi_2(p)$, ... is constructed according to the formula

$$\varphi_{j+1} = \frac{1}{T(p)} \int K(p|p')\varphi_j(p')dp' \quad (2.3)$$

Further the two sets of integrals are formed:

$$T_j = \int \varphi_j^*(p)T(p)\varphi_j(p)dp \quad (2.4)$$

$$\begin{aligned} \Phi_j &= \int \int \varphi_j^*(p)K(p|p')\varphi_j(p')dpdp' \\ j &= 0, 1, 2 \dots \end{aligned} \quad (2.5)$$

The function $\varphi_0(p)$ is chosen so that these integrals exist. For an arbitrary function $u(p)$ we have

$$T_u = \int u^*(p)T(p)u(p)dp \geq 0 \quad (2.6)$$

and

$$\Phi_u = \int \int u^*(p)K(p|p')u(p')dpdp' \geq 0 \quad (2.7)$$

according to (2.2). Further the successive ratios

$$\lambda_j = T_j / \Phi_j \quad (2.8)$$

$$\lambda_{j+\frac{1}{2}} = \Phi_j / T_{j+1} \quad (2.9)$$

are formed.

We shall now prove the following inequalities:

$$\lambda_j \geq \lambda_{j+\frac{1}{2}} \quad (2.10)$$

$$\lambda_{j+\frac{1}{2}} \geq \lambda_{j+1} \quad (2.11)$$

The proof runs exactly as the proof of Schwartz' inequality. We insert in (2.6) and (2.7)

$$u(p) = \alpha \varphi_j(p) + \beta \varphi_{j+1}(p) \quad (2.12)$$

where α and β are real, and find

$$\alpha^2 T_j + 2\alpha\beta\Phi_j + \beta^2 T_{j+1} \geq 0 \quad (2.13)$$

$$\alpha^2 \Phi_j + 2\alpha\beta T_{j+1} + \beta^2 \Phi_{j+1} \geq 0 \quad (2.14)$$

These inequalities are valid for all values of α and β including $\beta = 0$; as conditions for this, we have

$$T_j \geq 0 \quad \Phi_j \geq 0 \quad (2.15)$$

$$T_j T_{j+1} \geq \Phi_j^2 \quad (2.16)$$

$$\Phi_j \Phi_{j+1} \geq T_{j+1}^2 \quad (2.17)$$

or, according to (2.8) and (2.9),

$$\lambda_j \geq \lambda_{j+\frac{1}{2}} \quad (2.10)$$

$$\lambda_{j+\frac{1}{2}} \geq \lambda_{j+1} \quad (2.11)$$

Now the variational principle states that the lowest eigenvalue λ is a lower bound of the ratio

$$\lambda(u) = T_u / \Phi_u \quad (2.18)$$

for arbitrary functions $u(p)$. Especially with $u(p) = \varphi_j(p)$ we have

$$\lambda_j \geq \lambda \quad (2.19)$$

Hence the numbers

$$\lambda_0 \lambda_{\frac{1}{2}} \lambda_1 \dots \quad (\lambda)$$

constitute a sequence, which never increases or falls below the lowest eigenvalue λ .

Furthermore, it can be proved (cf. KELLOGG 1922) that the limit $\bar{\lambda}$ of the sequence (λ) is an eigenvalue of (2.1) and equal to the lowest eigenvalue λ in the case where $\varphi_0(p)$ satisfies the condition

$$\int \varphi_0^*(p) T(p) u(p) dp \neq 0 \quad (2.20)$$

where $u(p)$ is the exact eigenfunction corresponding to the lowest eigenvalue λ . In the latter case the sequence of functions

$\varphi_j(p)/\sqrt{T_j}$ ($j=0, 1, 2 \dots$) converges uniformly to the exact eigenfunction $u(p)$.

When the kernel is not positive-definite we come to the much weaker result

$$\lambda_j \geq \lambda_{j+\frac{1}{2}} \quad (2.21)$$

$$\lambda_j \lambda_{j+\frac{1}{2}} \geq \lambda_{j+1} \lambda_{j+\frac{3}{2}} \quad (2.22)$$

The principles of the above method are due to KELLOGG (1922). COLLATZ (1940) also gives methods by means of which lower bounds of the eigenvalues can be computed. Similar methods are proposed by THOMAS (1937).

The usefulness of the iteration method in its applications depends on the choice of the function $\varphi_0(p)$ and the number of integrals, T_j and Φ_j , which can be actually computed. To gain a starting position which is as favourable as possible, we assume that the function $\varphi_0(p)$ is a rather simple analytical function of the variables p , but that it depends upon at least one arbitrary parameter α which is so chosen that the quantity λ_0 , being a function of α , has a minimum with respect to α . The value of α is preserved in the following approximations $\lambda_{\frac{1}{2}}, \lambda_1 \dots$. The improvements which could be achieved by varying α in the latter quantities too are, as a rule, quite small. Thus the computation of λ_0 is completely equivalent to the variational method and subject to the same limitations as this method with regard to the choice of trial functions and convergence. The calculation of $\lambda_{\frac{1}{2}}$ (or $\lambda_{\frac{1}{2}}$ and λ_1) then represents definite progress. The improvements may be expected to be large whenever the variational method yields bad results, *e.g.* for short-range potentials. Compared with the complete (Ritz) variational method an advantage of the variation-iteration (V—I) method is that the approximate eigenvalues $\lambda_0, \lambda_{\frac{1}{2}}, \lambda_1$ are expressed *explicitly* and not as roots of secular equations.

3. The V—I Method Illustrated.

In order to illustrate the V—I method by an example, such a kernel is chosen that the integral equation can be solved exactly by means of known functions. We take the one-dimensional equation previously discussed by THOMAS (1937).

$$(p_0^2 + p^2) \varphi(p) = \frac{\lambda}{\pi} \int_{-\infty}^{+\infty} \frac{a^3}{a^2 + (p-p')^2} \varphi(p') dp' \quad (3.1)$$

It derives from the deuteron problem with an exponential potential.

We could put

$$\varphi_0(p) = \frac{p}{p^2 + b^2} \quad (3.2)$$

as a trial wave function. As the integral T_0 does not converge in this case, however, we may take the function

$$\varphi_0(p) = \frac{1}{p_0^2 + p^2} \cdot \frac{p}{p^2 + b^2} \quad (3.3)$$

which is mainly the iterate of the function (3.2). The integrals T_0 , Φ_0 (Equations 2.4₀, 2.5₀) can easily be evaluated by calculus of residues. The result is expressed by

$$\lambda_0 = \frac{T_0}{\Phi_0} = \frac{(a+b+p_0)(a+2b)(a+2p_0)}{4a^2b} \quad (3.4)$$

Minimization with respect to b gives

$$b^2 = \frac{1}{2} a(a+p_0) \quad (3.5)$$

To get a numerical example, we put $p_0 = 0$ and have

$$\lambda_0 = \frac{3+2\sqrt{2}}{4} \quad (3.6)$$

The next approximations are written only in this special case:

$$\lambda_{\frac{1}{2}} = \frac{4(a+b)^2(2a+b)}{(a+2b)(5a^2+5ab+b^2)} \quad (3.7)$$

$$\lambda_1 = \frac{3(3a+b)(3a+2b)(5a^2+5ab+b^2)}{2(a+b)(2a+b)(23a^2+13ab+2b^2)} \quad (3.8)$$

or, if the value of b (Equation 3.5) is inserted,

$$\lambda_0 = 1.457, \lambda_{\frac{1}{2}} = 1.4466, \lambda_1 = 1.44587.$$

The actual value of λ is the first root of the Bessel equation

$$J_0(2\sqrt{x})=0, \text{ which is } x=1.44580$$

We test this by solving the integral equation (3.1) directly. By means of calculus of residues we prove the formula

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{a^3}{a^2+(p-p')^2} \cdot \frac{p'}{p'^2+b^2} dp' = a^2 \cdot \frac{p}{p^2+(a+b)^2} \quad (3.9)$$

Accordingly, we attempt a formal solution

$$\varphi(p) = \sum_{v=0}^{\infty} c_v \frac{p}{p^2+(av+p_0)^2} \quad (3.10)$$

If this series is inserted in (3.1) and integrated term-by-term, we find

$$\sum_{v=0}^{\infty} c_v p \left[1 - \frac{(av+p_0)^2 - p_0^2}{p^2 + (av+p_0)^2} \right] = \lambda a^2 \sum_{v=1}^{\infty} c_{v-1} \frac{p}{p^2 + (av+p_0)^2} \quad (3.11)$$

which is an identity if

$$-c_v [(av+p_0)^2 - p_0^2] = \lambda a^2 c_{v-1} \quad (v=1, 2, \dots) \quad (3.12)$$

$$\sum_{v=0}^{\infty} c_v = 0 \quad (3.13)$$

From (3.12) we derive

$$\frac{c_v}{c_0} = \prod_{\mu=1}^v \frac{c_{\mu}}{c_{\mu-1}} = \frac{(-\lambda)^v}{v!} \prod_{\mu=1}^v \frac{1}{\mu + \frac{2p_0}{a}} \quad (3.14)$$

The condition (3.13) is then to be satisfied, whence

$$c_0 + c_0 \sum_{v=1}^{\infty} \frac{(-\lambda)^v}{v!} \frac{1}{\left(1 + \frac{2p_0}{a}\right) \dots \left(v + \frac{2p_0}{a}\right)} = 0 \quad (3.15)$$

or

$$\lambda^{-\frac{1}{2}q} J_q(2\sqrt{\lambda}) = 0 \quad q = \frac{2p_0}{a} \quad (3.16)$$

Evidently, for a fixed value of p the series (3.10) converges more rapidly than the series (3.13), which is essentially a Bessel series. As the convergence of (3.10) is uniform, the integration can be performed term-by-term. Thus the function $\varphi(p)$ defined by (3.10) and (3.14) is a solution of (3.1) if λ is a root of (3.16).

A closer investigation of the present problem shows that Equation (3.10) represents all odd solutions, while all even solutions are obtained in an analogous way.

CHAPTER II.

The Ground State of a Two-body System with the Error Function Potential.

Since the discovery of an electrical quadrupole moment of the deuteron, the view has been generally accepted that a part, sometimes even the major part, of the binding energy of the deuteron derives from an angle-dependent interaction similar in character to the interaction between two dipoles. It may be emphasized, however, that the contribution of such a force to the binding energy is strongly dependent on the ratio between the range of this force and that of the ordinary angle-independent force. In either case a treatment of the two-body problem for a central force of the error function type may be justified as being a preparatory study to the subsequent investigation.

4. Calculation of the Force Constant.

The potential energy in coordinate space is expressed by

$$V(|\mathbf{r}|) = -Be^{-r^2/a^2} \quad (4.1)$$

The kernel of the integral equation (1.12) can be computed explicitly. At the same time a unit of momentum is introduced so that $\hbar = h/2\pi = 1$. We then have

$$\begin{aligned} & \int \exp\{-i(\mathbf{p}-\mathbf{p}') \cdot \mathbf{r}\} V(|\mathbf{r}|) d\mathbf{r} = \\ & = -B \prod_{xyz} \int_{-\infty}^{+\infty} \exp\{-i(p_x - p'_x)x - x^2/a^2\} dx = \\ & = -B(a\sqrt{\pi})^3 \exp\left\{-\frac{a^2}{4}(\mathbf{p}-\mathbf{p}')^2\right\} \end{aligned} \quad (4.2)$$

Hence the integral equation (1.12) is written

$$(p_0^2 + p^2) \varphi(\mathbf{p}) = \lambda a (4\pi)^{-\frac{3}{2}} \int \exp \left\{ -\frac{a^2}{4} (\mathbf{p} - \mathbf{p}')^2 \right\} \varphi(\mathbf{p}') d\mathbf{p}' \quad (4.3)$$

with the abbreviations

$$p_0^2 = -ME \quad (4.4)$$

$$\lambda = BMa^2 \quad (4.5)$$

Wave functions. When the variational method is applied in coordinate space, a commonly employed trial function is

$$\psi_0(\mathbf{r}) = e^{-r^2/b^2} \quad (4.6)$$

This function is translated into momentum space:

$$\begin{aligned} \varphi_0(\mathbf{p}_1, \mathbf{p}_2) &= \int \int (\mathbf{p}_1 | \mathbf{r}_1) (\mathbf{p}_2 | \mathbf{r}_2) \exp \{ -(\mathbf{r}_1 - \mathbf{r}_2)^2 / b^2 \} d\mathbf{r}_1 d\mathbf{r}_2 = \\ &= (2\pi)^{-3} \int \int \exp \{ -i(\mathbf{p}_1 \cdot \mathbf{r}_1 + \mathbf{p}_2 \cdot \mathbf{r}_2) - (\mathbf{r}_1 - \mathbf{r}_2)^2 / b^2 \} d\mathbf{r}_1 d\mathbf{r}_2 = \\ &= \delta(\mathbf{p}_1 + \mathbf{p}_2) b^3 \pi^{\frac{3}{2}} \exp \left\{ -\frac{b^2}{4} p_1^2 \right\} \end{aligned} \quad (4.7)$$

Accordingly, we put

$$\varphi_0(\mathbf{p}) = \exp \left\{ -\frac{b^2}{4} p^2 \right\} \quad (4.8)$$

as an initial function in the V—I method.

Calculation of λ_0 . The quantities T_0 and Φ_0 are computed according to (2.4₀) and (2.5₀),

$$T_0 = \int (p_0^2 + p^2) \exp \left\{ -\frac{b^2}{2} p^2 \right\} d\mathbf{p} = (2\pi)^{\frac{3}{2}} \left(\frac{p_0^2}{b^3} + \frac{3}{b^5} \right) \quad (4.9)$$

$$\begin{aligned} \Phi_0 &= a (4\pi)^{-\frac{3}{2}} \int \int \exp \left\{ -\frac{a^2}{4} (\mathbf{p} - \mathbf{p}')^2 - \frac{b^2}{4} (p^2 + p'^2) \right\} d\mathbf{p} d\mathbf{p}' = \\ &= a (4\pi)^{-\frac{3}{2}} \prod_{p=p_x, p_y, p_z} \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dp' \exp \left\{ -\frac{1}{4} (a^2 + b^2) (p^2 + p'^2) + \frac{a^2}{2} p p' \right\} = \\ &= (4\pi)^{\frac{3}{2}} a b^{-3} (2a^2 + b^2)^{-\frac{3}{2}} \end{aligned} \quad (4.10)$$

using a well-known integration formula

$$\int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_n \exp \left\{ - \sum_1^n c_{jk} x_j x_k \right\} = \pi^{\frac{n}{2}} (\det c_{jk})^{-\frac{1}{2}} \quad (4.11)$$

where $\sum_1^n c_{jk} x_j x_k$ is a positive-definite quadratic form. This formula will be used frequently in the subsequent reasoning.

Now

$$\lambda_0 = \frac{T_0}{\Phi_0} = \frac{1}{a} \left(p_0^2 + \frac{3}{b^2} \right) \left(a^2 + \frac{b^2}{2} \right)^{\frac{3}{2}} \quad (4.12)$$

is minimized with respect to the parameter b^2 , which is determined as the positive root of

$$b^4 p_0^2 + b^2 - 4a^2 = 0 \quad (4.13)$$

The relation between λ_0 and p_0 is expressed in the parametric form

$$\lambda_0 = B_0 M a^2 = \frac{(k+1)^{\frac{5}{2}}}{k^{\frac{1}{2}}} \quad (4.14)$$

$$a^2 p_0^2 = -E M a^2 = k^2 - \frac{1}{2} k \quad (4.15)$$

where

$$k = \frac{2a^2}{b^2} \quad (4.16)$$

The dimensionless quantity $\lambda_0 = B_0 M a^2$ is plotted against a (in 10^{-13} cm) as the curve »O» in Figure 1. The energy of the ground state of the deuteron is 2.183 ± 0.012 Mev. (MYERS and VAN ATTA, 1942).

Calculation of $\lambda_{\frac{1}{3}}$. First the iterated function $\varphi_1(\mathbf{p})$ is computed

$$\begin{aligned} (p_0^2 + p^2) \varphi_1(\mathbf{p}) &= a (4\pi)^{-\frac{3}{2}} \int \exp \left\{ -\frac{a^2}{4} (\mathbf{p} - \mathbf{p}')^2 \right\} \varphi_0(\mathbf{p}') d\mathbf{p}' = \\ &= a (4\pi)^{-\frac{3}{2}} \prod_{p=\nu_x, \nu_y, \nu_z} \int_{-\infty}^{+\infty} dp' \exp \left\{ -\frac{a^2}{4} (p-p')^2 - \frac{b^2}{4} p'^2 \right\} = \\ &= \frac{a}{(a^2 + b^2)^{\frac{3}{2}}} \exp \left\{ -\frac{b^2}{4} \cdot \frac{a^2}{a^2 + b^2} p^2 \right\} \end{aligned} \quad (4.17)$$

Then T_1 defined by (2.4₁) can be evaluated:

$$\begin{aligned} T_1 &= \int (p_0^2 + p^2) [\varphi_1(\mathbf{p})]^2 d\mathbf{p} = \\ &= \frac{a^2}{(a^2 + b^2)^3} \int \frac{1}{p_0^2 + p^2} \exp \left\{ -\frac{b^2}{2} \cdot \frac{a^2}{a^2 + b^2} p^2 \right\} d\mathbf{p} \end{aligned} \quad (4.18)$$

If the factor $\frac{1}{p_0^2 + p^2}$ is written as an integral

$$\frac{1}{p_0^2 + p^2} = \int_0^\infty dt \exp \{ -(p_0^2 + p^2)t \} \quad (4.19)$$

the integration with respect to $\mathbf{p}$ can be performed for each coordinate axis and the result can be expressed by the error integral

$$\begin{aligned} T_1 &= \frac{a^2}{(a^2 + b^2)^3} \int_0^\infty dt \exp \{ -p_0^2 t \} \int \exp \left\{ -p^2 t - \frac{b^2}{2} \cdot \frac{a^2}{a^2 + b^2} p^2 \right\} d\mathbf{p} = \\ &= \frac{a^2 \pi^{\frac{3}{2}}}{(a^2 + b^2)^3} \int_0^\infty dt \exp \{ -p_0^2 t \} \left(t + \frac{b^2}{2} \cdot \frac{a^2}{a^2 + b^2} \right)^{-\frac{3}{2}} = \\ &= \frac{a^2 p_0 (2\pi)^{\frac{3}{2}}}{(a^2 + b^2)^3} \left[\frac{1}{x} - F(x) \right] \end{aligned} \quad (4.20)$$

where

$$x = ab p_0 (a^2 + b^2)^{-\frac{1}{2}} \quad (4.21)$$

and $F(x)$ is a function defined by

$$F(x) = \int_x^\infty \exp \left\{ \frac{x^2 - u^2}{2} \right\} du \quad (4.22)$$

and tabulated in the B. A. Mathematical Tables, Vol. VII, 1939. Thus we obtain, according to (2.9₀) and (4.10),

$$\lambda_{\frac{1}{2}} = \frac{\Phi_0}{T_1} = \frac{(a^2 + b^2)^3}{a p_0 b^3 \left(a^2 + \frac{1}{2} b^2 \right)^{\frac{3}{2}}} \left[\frac{1}{x} - F(x) \right]^1 \quad (4.23)$$

or, according to (4.5), (4.15) and (4.16),

$$\lambda_1 = B_1 Ma^2 = \frac{(k+2)^3}{2k^{\frac{1}{2}}(k+1)^{\frac{3}{2}}(2k-1)^{\frac{1}{2}}} \left[\frac{1}{x} - F(x) \right]^{-1} \quad (4.24)$$

$$x = \left(k \cdot \frac{2k-1}{k+2} \right)^{\frac{1}{2}} \quad (4.25)$$

The result is reproduced as the curve $\gg \frac{1}{2} \gg$ in Figure 1.

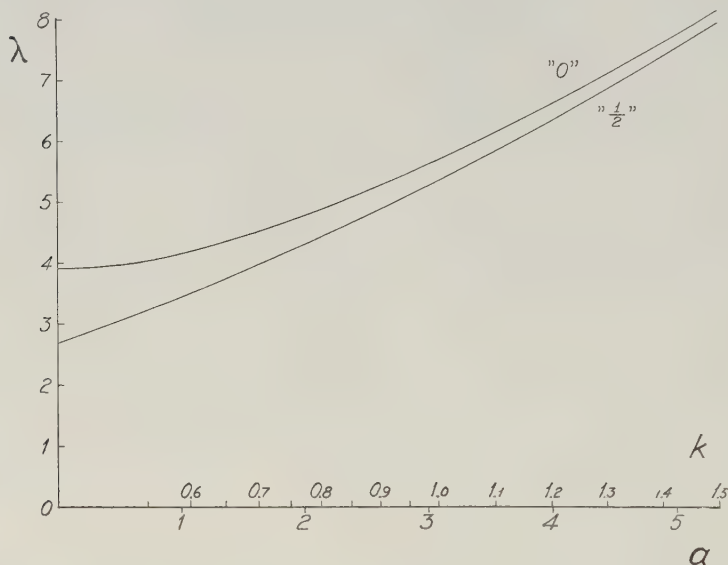


Figure 1. The force constant λ as a function of the range a (in 10^{-13} cm) and of the parameter k .

Calculation of λ_1 . We have according to (2.5₁) and (4.17)

$$\begin{aligned} \Phi_1 &= a(4\pi)^{-\frac{3}{2}} \int \int \varphi_1(\mathbf{p}) \exp \left\{ -\frac{a^2}{4} (\mathbf{p} - \mathbf{p}')^2 \right\} \varphi_1(\mathbf{p}') d\mathbf{p} d\mathbf{p}' = \\ &= \frac{a^3 (4\pi)^{-\frac{3}{2}}}{(a^2 + b^2)^3} \int \int \frac{1}{p_0^2 + p^2} \frac{1}{p_0'^2 + p'^2} \exp \left\{ -\frac{a^2}{4} (\mathbf{p} - \mathbf{p}')^2 - \frac{b'^2}{4} (p^2 + p'^2) \right\} d\mathbf{p} d\mathbf{p}' \end{aligned} \quad (4.26)$$

where

$$b'^2 = b^2 \cdot \frac{a^2}{a^2 + b^2}$$

We proceed in a manner similar to the above and make the substitution

$$\frac{1}{p_0^2 + p^2} \cdot \frac{1}{p_0'^2 + p'^2} = \int_0^\infty du \int_0^\infty dv \exp \{ -(p_0^2 + p^2)u - (p_0'^2 + p'^2)v \} \quad (4.27)$$

whereby the 6-dimensional integral (4.26) is transformed into an 8-dimensional integral, in which the integrations with respect to $\mathbf{p}$ and $\mathbf{p}'$ can be effected by means of (4.11). The result is a double integral

$$\Phi_1 = \frac{1}{8} \cdot \left(\frac{a\sqrt{\pi}}{a^2 + b^2} \right)^3 \int_0^\infty du \int_0^\infty dv \exp \{ -p_0'^2(u+v) \} \left[\left(\frac{a^2 + b'^2}{4} + u \right) \cdot \left(\frac{a^2 + b'^2}{4} + v \right) - \frac{a^4}{16} \right]^{-\frac{3}{2}} \quad (4.28)$$

One more integration is possible, namely with respect to $u - v$. After some simplifications we have

$$\Phi_1 = a \left(\frac{\sqrt{\pi}}{a^2 + b^2} \right)^3 \left[\left(\frac{a^2 + b'^2}{4} \right)^2 - \left(\frac{a^2}{4} \right)^2 \right]^{-\frac{1}{2}} \left[\frac{2a^2 + b'^2}{4} G_1 - \frac{b'^2}{4} \overline{G}_1 \right] \quad (4.29)$$

where

$$G_1 = G \left(\frac{2a^2 + b'^2}{b'^2}, \frac{b'^2 p_0'^2}{4} \cdot \frac{2a^2 + b'^2}{a^2 + b'^2} \right) \quad (4.30)$$

$$\overline{G}_1 = G \left(\frac{b'^2}{2a^2 + b'^2}, \frac{b'^2 p_0'^2}{4} \cdot \frac{2a^2 + b'^2}{a^2 + b'^2} \right) \quad (4.31)$$

$$G(\mu, \kappa) = \int_1^\infty e^{-\kappa t + \mu} \frac{1}{t + \mu} \cdot \frac{dt}{\sqrt{t}} \quad (4.32)$$

The integral (4.32) must be computed numerically by series expansion. The difference $\lambda_2' - \lambda_1$ is very small compared with $\lambda_0 - \lambda_1$. For values of a between 0 and $4 \cdot 10^{-13}$ cm it proves to be approximately 0.2 per cent of the value of λ_2' .

5. The Asymptotic Properties of the Eigenfunctions.

The most serious failure of the eigenfunction $\psi_0(\mathbf{r})$, Equation (4.6), in coordinate space is the false asymptotic behaviour for $r \rightarrow \infty$. The corresponding failure is inherent in the function $\varphi_0(\mathbf{p})$, Equation (4.8), in momentum space. We shall now prove that

this error is corrected in the iterated function $\varphi_1(\mathbf{p})$, defined by (4.17). We translate this function to coordinate space and find, neglecting some unimportant constants,

$$\begin{aligned}\psi_1(\mathbf{r}) &= \int \exp\{i\mathbf{p} \cdot \mathbf{r}\} \frac{1}{p_0^2 + p^2} \exp\left\{-\frac{\gamma^2}{2} p^2\right\} d\mathbf{p} = \\ &= \frac{2\pi}{ir} \int_{-\infty}^{+\infty} \exp\left\{ipr - \frac{\gamma^2}{2} p^2\right\} \frac{p dp}{p_0^2 + p^2}\end{aligned}\quad (5.1)$$

This integral can be evaluated by a parallel displacement of the integration path to the line $p = ir/\gamma^2$. Under the condition

$$\frac{r}{\gamma^2} > p_0 \quad (5.2)$$

the pole $p = ip_0$ is passed by. Hence

$$\begin{aligned}\psi_1(\mathbf{r}) &= \frac{2\pi^2}{r} \exp\left\{-p_0 r + \frac{\gamma^2}{2} p_0^2\right\} + \\ &+ \frac{2\pi}{ir} \int_{-\infty}^{+\infty} \exp\left\{-\frac{r^2}{2\gamma^2} - \frac{\gamma^2}{2} p^2\right\} \cdot \frac{p + \frac{ir}{\gamma^2}}{p_0^2 + \left(p + \frac{ir}{\gamma^2}\right)^2} dp\end{aligned}\quad (5.3)$$

The last term simplified is a real integral, namely

$$\frac{\pi}{r} \int_{-\infty}^{+\infty} \exp\left\{-\frac{r^2}{2\gamma^2} - \frac{\gamma^2}{2} p^2\right\} \left[\frac{\frac{r}{\gamma^2} + p_0}{p^2 + \left(\frac{r}{\gamma^2} + p_0\right)^2} + \frac{\frac{r}{\gamma^2} - p_0}{p^2 + \left(\frac{r}{\gamma^2} - p_0\right)^2} \right] dp \quad (5.4)$$

By means of the formula

$$\int_{-\infty}^{+\infty} \exp\{-c^2 p^2\} \frac{dp}{p^2 + q^2} = 2\sqrt{\pi} \int_c^{\infty} \exp\{q^2(c^2 - x^2)\} dx \quad (5.5)$$

the result can be expressed by error integrals

$$\begin{aligned}\psi_1(\mathbf{r}) &= \frac{2\pi^2}{r} \exp\left\{-p_0 r + \frac{\gamma^2}{2} p_0^2\right\} - \\ &\frac{\pi}{r} \sqrt{2\pi} \exp\left\{-\frac{r^2}{2\gamma^2}\right\} \left[F\left(\frac{r}{\gamma} + p_0 \gamma\right) + F\left(\frac{r}{\gamma} - p_0 \gamma\right) \right]\end{aligned}\quad (5.6)$$

The function $F(x)$ which is defined by (4.22), has the asymptotic expansion

$$F(x) = \frac{1}{x} - \frac{1}{x^3} + \frac{1 \cdot 3}{x^5} - + \dots \quad (5.7)$$

Hence the last term in (5.6) tends to zero faster than $\exp\left\{-\frac{r^2}{2\gamma^2}\right\}$ as r tends to infinity. The asymptotic character is then governed by the first term, which falls off exponentially:

$$\psi_1(\mathbf{r}) \rightarrow \text{const.} \frac{1}{r} e^{-p_0 r} \quad (5.8)$$

If $r/\gamma^2 < p_0$, the integration path in (5.1) can be displaced without passing the pole $p = ip_0$, the remaining calculation being analogous. The result is

$$\psi_1(\mathbf{r}) = \frac{\pi}{r} \sqrt{2\pi} \exp\left\{-\frac{r^2}{2\gamma^2}\right\} \left[F\left(p_0 \gamma - \frac{r}{\gamma}\right) - F\left(p_0 \gamma + \frac{r}{\gamma}\right) \right] \quad (5.9)$$

This representation of the wave function, valid in the region $0 \leq r \leq p_0 \gamma^2$, is an even function of r , as it should be in the vicinity of $r = 0$. Cf. WEIZSÄCKER (1937), p. 68.

In the calculation of the mean value of the total energy with the wave function $\psi_0(\mathbf{r})$ according to the ordinary variational method, the wrong asymptotic character of $\psi_0(\mathbf{r})$ is most manifested in the kinetic energy, while the error function in the potential energy term eliminates the influence of $\psi_0(\mathbf{r})$ in the range of large r -values. It may be remembered that the V—I method corrects in the first approximation ($\lambda_{\frac{1}{2}}$) only the kinetic energy (strictly speaking also the normalization integral).

6. On the Rapidity of Convergence. Comparison with the Ordinary Variational Method.

The present problem has been treated by several authors by means of different methods and from various points of view. A numerical solution of the differential equation has been presented by FEENBERG (1935) (cf. B B, p. 111). Other authors have used the ordinary variational method with variation functions chosen from different starting-points. In this connexion we shall

direct our attention especially upon the work of WARREN and MARGENAU (1937). In an earlier paper (MARGENAU and WARREN 1937) these authors have investigated the three- and four-body problems very closely. The calculation of the two-body problem has been made by the same method in order to provide a criterion of the reliability of the results from the calculations on the previous problems. Their method is as follows. With the variation function

$$\psi_0(\mathbf{r}) = e^{-r^2/b^2} \quad (6.1)$$

the energy is calculated and minimized with respect to b (the *scale parameter*). Then the trial function (6.1) is extended by means of a linear set of symmetrized oscillator wave functions

$$\psi(\mathbf{r}) = \sum_{lmn} c_{lmn} \psi_{lmn}(\mathbf{r}) \quad (6.2)$$

The function $\psi_{lmn}(\mathbf{r})$ represents all symmetric wave functions of the three-dimensional harmonic oscillator belonging to the total quantum number $l+m+n$ and can be expressed by one-dimensional oscillator functions according to the formula

$$\psi_{lmn}(\mathbf{r}) = N_{lmn} (\varphi_{lmn} + \varphi_{lnm} + \varphi_{mln} + \varphi_{mnl} + \varphi_{nlm} + \varphi_{nml}) \quad (6.3)$$

where N_{lmn} is a normalization factor

$$N_{lmn} = \frac{1}{\sqrt{6}} (1 + \delta_{lm} + \delta_{ln} + \delta_{mn} + 2\delta_{lm}\delta_{ln}\delta_{mn})^{-\frac{1}{2}} \quad (6.4)$$

The functions φ_{lmn} are defined by

$$\varphi_{lmn} = \varphi_l(x) \varphi_m(y) \varphi_n(z) \quad (6.5)$$

$$\varphi_l(x) = N_l e^{-x^2/2} H_l(x) \quad (6.6)$$

where $H_l(x)$ is an Hermite polynomial.

The trial function (6.1) is the first term of the series (6.2). With the value of the parameter b in (6.1) preserved in (6.2) the coefficients c_{lmn} are varied in order to make the energy a minimum. If the eigenvalue problem is proposed in momentum space and a corresponding trial function is assumed, the results will be identical. As is well known, the oscillator functions in coordinate space correspond to oscillator functions in momentum space. It appears from the calculations of Warren and Margenau (Table II, p. 1028) that the energy converges very slowly towards

the true limit with increasing numbers of oscillator functions introduced. Matters are the same if the force constant λ_0 is calculated for a fixed value of the energy as is shown in our Table I. In their discussion of the results Warren and Margenau emphasize especially the importance of *cooperation* of oscillator functions corresponding to different degrees of excitation for obtaining good results. The functions $\psi_{lmn}(\mathbf{r})$ of (6.2) have the same wrong asymptotical character for $r \rightarrow \infty$ as the single function (6.1), and we may well ask whether the method of linear sets of variation functions can be combined with the iteration method, or, more precisely, to what extent the results of the iteration method can be improved if a function $\varphi_0(\mathbf{p})$ corresponding to the function (6.2) is chosen as an initial function. To obtain an answer to this question we put

$$\varphi_0(\mathbf{p}) = \sum_L c_L \psi_L(\mathbf{p}) \quad (6.7)$$

where the letter L represents the three indices $l m n$. Now it is evident that the coefficients c_L can be determined so that $\lambda_0 = T_0(c_L) / \Phi_0(c_L)$ has a minimum and then be inserted into the expression

$$\lambda_{\frac{1}{2}} = \Phi_0(c_L) / T_1(c_L) \quad (6.8)$$

It is, however, more practical and even more refined to determine the coefficients c_L directly by requiring the quantity $\lambda_{\frac{1}{2}}$ to have a minimum with respect to these coefficients. The quantities $\Phi_0(c_L)$ and $T_1(c_L)$ are quadratic forms in the numbers c_L

$$\Phi_0(c_L) = \sum_{L,L'} c_L c_{L'} \Phi_0(L|L') \quad (6.9)$$

$$T_1(c_L) = \sum_{L,L'} c_L c_{L'} T_1(L|L') \quad (6.10)$$

whence the minimum of $\lambda_{\frac{1}{2}}$ is obtained as the smallest root of the secular equation

$$|\Phi_0(L|L') - \lambda_{\frac{1}{2}} T_1(L|L')| = 0 \quad (6.11)$$

which is analogous to the secular equation by means of which λ_0 is determined:

$$|T_0(L|L') - \lambda_0 \Phi_0(L|L')| = 0 \quad (6.12)$$

The matrix elements $T_0(lmn|l'm'n')$ and $\Phi_0(lmn|l'm'n')$ are identical with the corresponding elements in coordinate space. The calculation of the matrix elements $T_1(lmn|l'm'n')$ is presented in Appendix I.

As is seen from Table I, the eigenvalue $\lambda_{\frac{1}{2}}$ cannot be improved to any appreciable degree if doubly and quadruply excited states are included. A comparison between Tables (a) and (b) shows that the correction of $\lambda_{\frac{1}{2}}$ is smaller for smaller values of k , that is for forces of shorter range.

TABLE I.

a) $k=0.6$ b) $k=0.5$

Functions employed	λ_0	$\lambda_{\frac{1}{2}}$	λ_1	λ_0	$\lambda_{\frac{1}{2}}$	λ_1	$\lambda_{\frac{3}{2}}$
(000)	4.180	3.491	3.481	3.897	2.690	2.6843	2.68406
(000) (200)	4.180	3.487		3.897	2.690		
(000) (400) (220)	3.840	3.484		3.517	2.689		
(000) (200) (400) (220)	3.713	3.480		3.353	2.685		

CHAPTER III.

The Nuclear Many-body Problems.

7. The Integral Equation of Many-body Problems.

If there be some doubt as to the true nature of the attractive force between the proton and the neutron in a deuteron, the uncertainty is not smaller with regard to the forces in a nucleus consisting of three or more nucleons. It is, indeed, known from scattering experiments that there exists an attractive force between two protons which is exactly (or very nearly exactly) as large as the force between a proton and a neutron in the same state (^1S), and even if the assumption that the force between two nucleons is the same for all couples $p-p$, $p-n$, $n-n$ is correct, the possibility cannot be excluded *a priori* that the force between two nucleons may be dependent on the position or the spin state of a third particle in the nucleus. On the contrary, most field theories, at least when the ordinary perturbation method is used, yield so-called many-body forces (JÁNOSY 1939, PRIMAKOFF and HOLSTEIN 1939). Such forces depend in a very complex manner upon the coordinates of a number of particles. Whether their appearance is a result of the divergences of the perturbation method or has a real meaning, is an open question. Before this question has been decided, we should keep to a more descriptive disposition: Are two-body forces sufficient to account for the binding energies and the scattering data or, if they are not, what may be put in their place?

As we aim to apply the same methods of calculation as above to a nucleus containing three or four nucleons, we first expound the many-body problem in momentum space. When only two-body

forces are admitted, the potential energy of a nucleus consisting of N nucleons can be written as a sum

$$V(\mathbf{r}_1 \dots \mathbf{r}_N) = V = \sum_{j < k} V_{jk}(\mathbf{r}_{jk}) \quad (7.1)$$

where $\mathbf{r}_{jk} = \mathbf{r}_j - \mathbf{r}_k$ denotes a vector pointing from the nucleon k to the nucleon j . To start with $V_{jk}(\mathbf{r}_{jk})$ may depend upon the spin vectors $\boldsymbol{\sigma}_j$ and $\boldsymbol{\sigma}_k$ in a manner not more closely defined. Precisely as in the case of the deuteron, the Schrödinger equation translated to momentum space may be written as an integral equation

$$\begin{aligned} \frac{1}{2M} \sum_{j=1}^N p_j^2 \psi(\mathbf{p}_1 \dots \mathbf{p}_N) + \int \dots \int (\mathbf{p}_1 \dots \mathbf{p}_N | V | \mathbf{p}'_1 \dots \mathbf{p}'_N) \psi(\mathbf{p}'_1 \dots \\ \dots \mathbf{p}'_N) d\mathbf{p}'_1 \dots d\mathbf{p}'_N = E \psi(\mathbf{p}_1 \dots \mathbf{p}_N) \end{aligned} \quad (7.2)$$

where

$$\begin{aligned} (\mathbf{p}_1 \dots \mathbf{p}_N | V | \mathbf{p}'_1 \dots \mathbf{p}'_N) = \int \dots \int (\mathbf{p}_1 | \mathbf{r}_1) \dots (\mathbf{p}_N | \mathbf{r}_N) V(\mathbf{r}_1 \dots \mathbf{r}_N) (\mathbf{r}_1 | \mathbf{p}'_1) \dots \\ \dots (\mathbf{r}_N | \mathbf{p}'_N) d\mathbf{r}_1 \dots d\mathbf{r}_N = \sum_{j < k} \int \int (\mathbf{p}_j | \mathbf{r}_j) (\mathbf{p}_k | \mathbf{r}_k) V_{jk}(\mathbf{r}_{jk}) \cdot \\ \cdot (\mathbf{r}_j | \mathbf{p}'_j) (\mathbf{r}_k | \mathbf{p}'_k) d\mathbf{r}_j d\mathbf{r}_k \cdot \prod_{\substack{l \neq j \\ l \neq k}} \delta(\mathbf{p}_l - \mathbf{p}'_l) \end{aligned} \quad (7.3)$$

If the latter expression is inserted in (7.2) and the integrations with respect to $\mathbf{p}_l$ ($l \neq j$, $l \neq k$) are performed, we obtain

$$\begin{aligned} \frac{1}{2M} \sum_{j=1}^N p_j^2 \psi(\mathbf{p}_1 \dots \mathbf{p}_N) + \sum_{j < k} \int \int (\mathbf{p}_j \mathbf{p}_k | V_{jk} | \mathbf{p}'_j \mathbf{p}'_k) \psi(\mathbf{p}_1 \dots \mathbf{p}'_j \dots \mathbf{p}'_k \dots \mathbf{p}_N) \cdot \\ \cdot d\mathbf{p}'_j d\mathbf{p}'_k = E \psi(\mathbf{p}_1 \dots \mathbf{p}_N) \end{aligned} \quad (7.4)$$

which is a kind of mixed integral equation or better a *partial* integral equation. In the second term of (7.4) the summation is extended to all pairs (j, k) of nucleons in interaction, while the integrations in each term of the double sum only concern the $\mathbf{p}'$ -coordinates of the two nucleons. Equation (7.4) is identical with Equation (7.2), the former being written formally as an ordinary integral equation. As V depends only on the relative coordinates $\mathbf{r}_{jk}$, one more integration can be carried out. Exactly as in the case of the deuteron, Equation (1.7), we have

$$\begin{aligned}
& (\mathbf{p}_j \mathbf{p}_k | V_{jk} | \mathbf{p}'_j \mathbf{p}'_k) = \\
& = \int \int (\mathbf{p}_j | \mathbf{r}_j) (\mathbf{p}_k | \mathbf{r}_k) V_{jk}(\mathbf{r}_{jk}) (\mathbf{r}_j | \mathbf{p}'_j) (\mathbf{r}_k | \mathbf{p}'_k) \cdot d\mathbf{r}_j d\mathbf{r}_k = \\
& = (2\pi)^{-3} \delta(\mathbf{p}_j + \mathbf{p}_k - \mathbf{p}'_j - \mathbf{p}'_k) \int \exp \{ -i(\mathbf{p}_j - \mathbf{p}'_j) \cdot \mathbf{r} \} V_{jk}(\mathbf{r}) d\mathbf{r} \quad (7.5)
\end{aligned}$$

Then Equation (7.4) is simplified into

$$\begin{aligned}
& \frac{1}{2M} \sum_{j=1}^N p_j^2 \psi(\mathbf{p}_1 \dots \mathbf{p}_N) + \sum_{j < k}^N \int U_{jk}(\mathbf{p}_j - \mathbf{p}'_j) \psi(\mathbf{p}_1 \dots \mathbf{p}'_{(j)} \dots \mathbf{p}_j + \\
& + \mathbf{p}_k - \mathbf{p}'_{(k)} \dots \mathbf{p}_N) d\mathbf{p}'_j = E \psi(\mathbf{p}_1 \dots \mathbf{p}_N) \quad (7.6)
\end{aligned}$$

where

$$U_{jk}(\mathbf{p}_j - \mathbf{p}'_j) = (2\pi)^{-3} \int \exp \{ -(\mathbf{p}_j - \mathbf{p}'_j) \cdot \mathbf{r} \} V_{jk}(\mathbf{r}) d\mathbf{r} \quad (7.7)$$

The equation (7.6) has a continuously infinite number of solutions since the coordinates used $\mathbf{p} = \{p_x, p_y, p_z\}$ are absolute. We must refer all coordinates to the motion of the centre of gravity. Since the latter can be assumed to be at rest, the solutions of (7.6) are written

$$\psi(\mathbf{p}_1 \dots \mathbf{p}_N) = \delta(\mathbf{p}_1 + \dots + \mathbf{p}_N) \varphi(\mathbf{p}_1 \dots \mathbf{p}_{N-1}) \quad (7.8)$$

Now both members of Equation (7.6) are integrated with respect to $\mathbf{p}_N$, the result being

$$\begin{aligned}
& \frac{1}{M} \left\{ \sum_{j=1}^{N-1} p_j^2 + \sum_{j < k}^{N-1} \mathbf{p}_j \cdot \mathbf{p}_k \right\} \varphi(\mathbf{p}_1 \dots \mathbf{p}_{N-1}) + \sum_{j < k}^{N-1} \int U_{jk}(\mathbf{p}_j - \mathbf{p}'_j) \cdot \\
& \cdot \varphi(\mathbf{p}_1 \dots \mathbf{p}'_{(j)} \dots \mathbf{p}_j + \mathbf{p}_k - \mathbf{p}'_{(k)} \dots \mathbf{p}_{N-1}) d\mathbf{p}'_j + \sum_{j=1}^{N-1} \int U_{jN}(\mathbf{p}_j - \mathbf{p}'_j) \cdot \\
& \cdot \varphi(\mathbf{p}_1 \dots \mathbf{p}'_{(j)} \dots \mathbf{p}_{N-1}) d\mathbf{p}'_j = E \varphi(\mathbf{p}_1 \dots \mathbf{p}_{N-1}) \quad (7.9)
\end{aligned}$$

8. On Nuclear Forces of Infinitely Small Range.

In the early discussion of the binding energies of the lightest nuclei, the extreme case of infinitely short-range nuclear forces was treated by several authors. According to the review by BETHE-BACHER the decisive step was taken by THOMAS (1935), who claimed to have shown that the assumption of an infinitely small range for the potential energy of a proton and a neutron in the triton nucleus was not compatible with the finite binding energy of the

triton. As Thomas's proof is rather involved, we take the occasion to discuss the same problem in momentum space.

The integral equation for a nucleus consisting of $N + 1$ nucleons with forces between all pairs of nucleons is written

$$T(\mathbf{p}_1 \dots \mathbf{p}_N) \varphi(\mathbf{p}_1 \dots \mathbf{p}_N) = \sum_{j < k}^N \int U_{jk}(\mathbf{p}_j - \mathbf{p}'_j) \varphi(\mathbf{p}_1 \dots \mathbf{p}'_{(j)} \dots \mathbf{p}_j + \mathbf{p}_k - \mathbf{p}'_j \dots \mathbf{p}_N) \cdot d\mathbf{p}'_j + \sum_{j=1}^N \int U_{j, N+1}(\mathbf{p}_j - \mathbf{p}'_j) \varphi(\mathbf{p}_1 \dots \mathbf{p}'_j \dots \mathbf{p}_N) d\mathbf{p}'_j \quad (8.1)$$

where

$$T(\mathbf{p}_1 \dots \mathbf{p}_N) = p_0^2 + \sum_{j=1}^N p_j^2 + \sum_{j < k}^N \mathbf{p}_j \cdot \mathbf{p}_k, \quad p_0^2 = -ME \quad (8.2)$$

If the range of the forces is infinitely small, we have

$$U_{jk}(\mathbf{p}) = \text{const.} = U_{jk} \quad (8.3)$$

and Equation (8.1) can be written

$$T(\mathbf{p}_1 \dots \mathbf{p}_N) \varphi(\mathbf{p}_1 \dots \mathbf{p}_N) = \sum_{j < k}^N \chi_{jk}(\mathbf{p}) + \sum_{j=1}^N \chi_j(\mathbf{p}) \quad (8.4)$$

where

$$\begin{aligned} \chi_{jk}(\mathbf{p}) &= \chi_{jk}(\mathbf{p}_1 \dots \mathbf{p}_{j-1} \mathbf{p}_{j+1} \dots \mathbf{p}_{k-1} \mathbf{p}_{k+1} \dots \mathbf{p}_N; \mathbf{p}_j + \mathbf{p}_k) = \\ &= U_0 \int \varphi(\mathbf{p}_1 \dots \mathbf{p}'_j \dots \mathbf{p}_j + \mathbf{p}_k - \mathbf{p}'_j \dots \mathbf{p}_N) d\mathbf{p}'_j \end{aligned} \quad (8.5)$$

$$\chi_j(\mathbf{p}) = \chi_j(\mathbf{p}_1 \dots \mathbf{p}_{j-1} \mathbf{p}_{j+1} \dots \mathbf{p}_N) = U_0 \int \varphi(\mathbf{p}_1 \dots \mathbf{p}'_j \dots \mathbf{p}_N) d\mathbf{p}'_j \quad (8.6)$$

if we assume all $U_{jk} = U_0$. The function $\chi_{jk}(\mathbf{p})$ depends on $\mathbf{p}_j$ and $\mathbf{p}_k$ only in the combination $\mathbf{p}_j + \mathbf{p}_k$, while $\chi_j(\mathbf{p})$ is independent of $\mathbf{p}_j$.

If $\varphi(\mathbf{p}_1 \dots \mathbf{p}_N)$ according to (8.4) is inserted in (8.6), for example, the function $\chi_j(\mathbf{p})$ will contain one term of the form

$$U_0 \int \frac{\chi_i(\mathbf{p})}{T(\mathbf{p}_1 \dots \mathbf{p}'_j \dots \mathbf{p}_N)} d\mathbf{p}'_j \quad (8.7)$$

This expression diverges because of the denominator, the nominator being independent of the integration variable $\mathbf{p}'_j$. Under these circumstances the problem has no solution for $U_0 \neq 0$. This result is almost trivial, but we may realize nevertheless that it

cannot be reproduced by a limiting process from a potential function of arbitrarily small range.

If we assume a potential energy

$$V(r) = Af\left(\frac{r}{a}\right) \text{ for } r < a$$

$$V(r) = 0 \quad \text{for } r \geq a$$
(8.8)

in coordinate space, where a is the range, we may write the corresponding term in momentum space

$$Aa^3 \int F(a|\mathbf{p} - \mathbf{p}'|) \varphi(\dots \mathbf{p}' \dots) d\mathbf{p}'$$
(8.9)

By introducing new coordinates $\mathbf{p}_j = \bar{\mathbf{p}}_j / a$, our integral equation (8.1) can be written

$$\begin{aligned} \bar{T}(\bar{\mathbf{p}}_1 \dots \bar{\mathbf{p}}_N) \bar{\varphi}(\bar{\mathbf{p}}_1 \dots \bar{\mathbf{p}}_N) = & Aa^2 \sum_{j < k}^N \int F_{jk}(\bar{\mathbf{p}}_j - \bar{\mathbf{p}}'_j) \bar{\varphi}(\dots \bar{\mathbf{p}}'_j \dots \bar{\mathbf{p}}_j + \bar{\mathbf{p}}_k - \\ & - \bar{\mathbf{p}}'_j \dots) d\bar{\mathbf{p}}'_j + Aa^2 \sum_{j=1}^N \int F_{j, N+1}(\bar{\mathbf{p}}_j - \bar{\mathbf{p}}'_j) \bar{\varphi}(\dots \bar{\mathbf{p}}'_j \dots) d\bar{\mathbf{p}}'_j \end{aligned}$$
(8.10)

where

$$\bar{T}(\bar{\mathbf{p}}_1 \dots \bar{\mathbf{p}}_N) = a^2 p_0^2 + \sum_{j=1}^N \bar{p}_j^2 + \sum_{j < k}^N \bar{\mathbf{p}}_j \cdot \bar{\mathbf{p}}_k$$
(8.11)

In general, for a fixed value of $a = a^2 p_0^2$, there exists a discrete set of eigenvalues of $\lambda = Aa^2$. If p_0^2 corresponds to a definite energy state, we have to show that the lowest eigenvalue of λ is a positive quantity, the value of the range a being arbitrarily small. This statement is equivalent to saying that λ or A for a fixed value of a has a definite positive value for a nucleus with an arbitrarily small binding energy p_0^2 .

Now, if an eigenfunction of Equation (8.10) is found for a definite value of a , we may write

$$\lambda(a) = \frac{T(a)}{\Phi(a)} \text{ or } \lambda(a) = \frac{a N_a + T'_a}{\Phi_a}$$
(8.12)

where N_a is the normalization integral and T'_a the kinetic energy term. The subscript a denotes that the dependence of a is an

implicit one. If an upper bound of $\lambda(\alpha)$ is calculated by means of the eigenfunction for $\alpha \rightarrow 0$,

$$\lambda(\alpha) \leq \frac{\alpha N_{\alpha=0} + T'_{\alpha=0}}{\Phi_{\alpha=0}} \quad (8.13)$$

we find that $\Phi_{\alpha=0}$ is bounded. Since $T'_{\alpha=0}$ is a positive quantity, we conclude that $\lambda(0) = T'_{\alpha=0} / \Phi_{\alpha=0}$ is positive. This result is contrary to that of Thomas, which seems to have been due to a false conclusion.

It can also be shown that $\lambda(0)$ represents the lowest value of $\lambda(\alpha)$. If an upper bound of $\lambda(0)$ is calculated with an eigenfunction for $\alpha \neq 0$, we have

$$\lambda(0) \leq \frac{T'_\alpha}{\Phi_\alpha} = \lambda(\alpha) - \alpha \frac{N_\alpha}{\Phi_\alpha} < \lambda(\alpha) \quad (8.14)$$

In an analogous way we may prove that $\lambda(\alpha)$ increases monotonously with increasing α .

9. The V—I Method.

The manner of constructing iterated functions in the equation (7.9) is obvious. The initial function is introduced in all terms containing integrals, while the resulting function has the coefficient

$$T(\mathbf{p}) = p_0^2 + \sum_{j=1}^{N-1} p_j^2 + \sum_{j < k}^{N-1} \mathbf{p}_j \cdot \mathbf{p}_k \quad (9.1)$$

where

$$p_0^2 = -ME$$

The quantity $T(\mathbf{p})$ is positive for all values of $\mathbf{p}_j$, if the total energy E is negative. In the following reasoning E means always the energy in the ground state of the nucleus. The method expounded in Section 2 for the calculation of the force constant can be taken over step by step. The integral sign in the definition of the entities T_j , Eq. (2.4), as well as the first integral sign of Φ_j , Eq. (2.5), represents integration with respect to the momentum coordinates $\mathbf{p}_1 \dots \mathbf{p}_{N-1}$. The only question arising concerns the definiteness of the whole expression corresponding to

the potential energy. At least in so far as every term corresponds to a positive-definite kernel, it is quite certain that the total expression is positive-definite, since we have for a potential energy term (j, k) of (7.9) integrated with respect to $\mathbf{p}_j$ and $\mathbf{p}_k$

$$\int \int \varphi^*(\mathbf{p}_1 \dots \mathbf{p}_j \dots \mathbf{p}_k \dots \mathbf{p}_{N-1}) \int U_{jk}(\mathbf{p}_j - \mathbf{p}'_j) \varphi(\mathbf{p}_1 \dots \mathbf{p}'_j \dots \mathbf{p}_j + \mathbf{p}_k - \mathbf{p}'_k \dots \mathbf{p}_{N-1}) \cdot d\mathbf{p}'_j d\mathbf{p}_j d\mathbf{p}_k \geq 0 \quad (9.2)$$

and, correspondingly,

$$\int \varphi^*(\mathbf{p}_1 \dots \mathbf{p}_j \dots \mathbf{p}_{N-1}) \int U_{jN}(\mathbf{p}_j - \mathbf{p}'_j) \varphi(\mathbf{p}_1 \dots \mathbf{p}'_j \dots \mathbf{p}_{N-1}) d\mathbf{p}'_j d\mathbf{p}_j \geq 0 \quad (9.3)$$

The last equation, which is valid for arbitrary functions φ of $\mathbf{p}_j$, is self-evident, the first also after the introduction of new variables

$$\begin{aligned} \mathbf{p}_j &= \mathbf{q}_{jk} + \frac{1}{2} \mathbf{p}_{jk} \\ \mathbf{p}'_j &= \mathbf{q}_{jk} + \frac{1}{2} \mathbf{p}'_{jk} \\ \mathbf{p}_k &= \mathbf{q}_{jk} - \frac{1}{2} \mathbf{p}_{jk} \end{aligned} \quad (9.4)$$

whence

$$\begin{aligned} & \int \int \int \varphi^* \left(\mathbf{p}_1 \dots \mathbf{q}_{jk} + \frac{1}{2} \mathbf{p}_{jk} \dots \mathbf{q}_{jk} - \frac{1}{2} \mathbf{p}_{jk} \dots \mathbf{p}_{N-1} \right) U_{jk} \left(\frac{1}{2} \mathbf{p}_{jk} \quad \frac{1}{2} \mathbf{p}'_{jk} \right) \cdot \\ & \cdot \varphi \left(\mathbf{p}_1 \dots \mathbf{q}_{jk} + \frac{1}{2} \mathbf{p}'_{jk} \dots \mathbf{q}_{jk} \quad \frac{1}{2} \mathbf{p}'_{jk} \dots \mathbf{p}_{N-1} \right) d\mathbf{p}_{jk} d\mathbf{p}'_{jk} d\mathbf{q}_{jk} \geq 0 \end{aligned} \quad (9.5)$$

valid for arbitrary functions φ of $\mathbf{p}_{jk}$.

The positive-definiteness is important in those cases where the accomplishment of higher approximations than the first ($\lambda_{\frac{1}{2}}$) is prohibited by computing difficulties.

If the forces depend on the spin vectors $\boldsymbol{\sigma}_j$, our scheme is essentially the same. Then the spin coordinates in the wave functions cannot be split off, so that one has, as a matter of fact, a system of partial integral equations. Summation with respect to the spin coordinates accompanies the integration over the momentum coordinates; for the rest, the manner of procedure and the conclusions are the same.

CHAPTER IV.

The Ground States of the Nuclei H^3 , He^3 and He^4 . Error Function Potential.

10. Calculation of the Force Constant for the Triton.

We assume that the forces between the three nucleons in the triton are equal and independent of the isotopic states of the nucleons, while the dependence on the spin is included by putting

$$V_{jk}(\mathbf{r}) = -B(1 - g + g P_{jk}) e^{-r^2/a^2} \quad (10.1)$$

for the potential energy of two nucleons at a distance r . The symbol P_{jk} denotes the spin exchange operator and g indicates the relative strength of the spin-dependent force. The integral equation of the triton is written

$$\begin{aligned} & (p_0^2 + p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2) \varphi(\mathbf{p}_1 \mathbf{p}_2 | 123) = \\ & = \lambda a (4\pi)^{-\frac{3}{2}} \left[\int (1 - g + g P_{12}) \exp \left\{ -\frac{a^2}{4} (\mathbf{p}_1 - \mathbf{p}'_1)^2 \right\} \varphi(\mathbf{p}'_1, \mathbf{p}_1 + \mathbf{p}_2 - \right. \\ & \quad \left. - \mathbf{p}'_1 | 123) d\mathbf{p}'_1 + \int (1 - g + g P_{13}) \exp \left\{ -\frac{a^2}{4} (\mathbf{p}_1 - \right. \right. \\ & \quad \left. \left. - \mathbf{p}'_1)^2 \right\} \varphi(\mathbf{p}'_1, \mathbf{p}_2 | 123) d\mathbf{p}'_1 + \int (1 - g + \right. \\ & \quad \left. + g P_{23}) \exp \left\{ -\frac{a^2}{4} (\mathbf{p}_2 - \mathbf{p}'_2)^2 \right\} \varphi(\mathbf{p}_1, \mathbf{p}'_2 | 123) d\mathbf{p}'_2 \right] \quad (10.2) \end{aligned}$$

where

$$p_0^2 = -ME \quad (10.3)$$

$$\lambda = B M a^2 \quad (10.4)$$

Wave functions. The spin coordinates upon which the operators P_{jk} act as permutations are denoted by $|123\rangle$. In applying the V—I method it is convenient to pay no regard to the spin forces in the initial wave function. If no spin forces were present, the wave function for the ground state would be antisymmetrical with respect to an interchange of the two neutrons 1 and 2.

Accordingly we put

$$\varphi_0(\mathbf{p}_1\mathbf{p}_2|123)=\chi_0(\mathbf{p}_1\mathbf{p}_2)\frac{1}{\sqrt{2}}[a(1)\beta(2)-a(2)\beta(1)]a(3) \quad (10.5)$$

employing the customary notation. The action of the part of the potential energy dependent on the spin presents itself in the iterated function.

A trial function frequently used in earlier investigations (*e. g.* FLÜGGE, 1937) of the triton problem is

$$\psi(\mathbf{r}_1\mathbf{r}_2\mathbf{r}_3)=\exp\{-(r_{12}^2+r_{13}^2+r_{23}^2)/b^2\} \quad (10.6)$$

A simple computation shows that it corresponds in momentum space to a function of the form

$$\chi_0(\mathbf{p}_1\mathbf{p}_2)=\exp\left\{-\frac{b^2}{4}(p_1^2+p_2^2+\mathbf{p}_1\cdot\mathbf{p}_2)\right\} \quad (10.7)$$

Instead of using explicit spin functions, we prefer to use mean values $\overline{f(P)}$ over functions f of the spin operators P . Only the following mean values are needed

$$\begin{aligned} \bar{P}_{12} &= -1, & \bar{P}_{13} &= \bar{P}_{23} = \frac{1}{2} \\ \overline{P_{12}^2} &= \overline{P_{13}^2} = \overline{P_{23}^2} = 1 \\ \overline{P_{12}P_{23}} &= \overline{P_{23}P_{31}} = \overline{P_{31}P_{12}} = -\frac{1}{2} \end{aligned} \quad (10.8)$$

The iterated function $\chi_1(\mathbf{p}_1\mathbf{p}_2)$ is then computed. We find

$$\begin{aligned} (p_0^2+p_1^2+p_2^2+\mathbf{p}_1\cdot\mathbf{p}_2)\chi_1(\mathbf{p}_1\mathbf{p}_2) &= \frac{a}{(a^2+b^2)^{\frac{3}{2}}}[(1-g+gP_{12})\omega_{12}(\mathbf{p}_1\mathbf{p}_2) + \\ &+ (1-g+gP_{13})\omega_{13}(\mathbf{p}_1\mathbf{p}_2) + (1-g+gP_{23})\omega_{23}(\mathbf{p}_1\mathbf{p}_2)] \end{aligned} \quad (10.9)$$

where

$$\begin{aligned} \omega_{12}(\mathbf{p}_1\mathbf{p}_2) &= \exp\left\{-\frac{b^2}{4}\cdot\frac{a^2}{a^2+b^2}(p_1^2+p_2^2+\mathbf{p}_1\cdot\mathbf{p}_2)-\right. \\ &\quad \left.-\frac{3b^4}{16}\cdot\frac{1}{a^2+b^2}(\mathbf{p}_1+\mathbf{p}_2)^2\right\} \end{aligned} \quad (10.10)$$

$$\omega_{13}(\mathbf{p}_1\mathbf{p}_2)=\omega_{23}(\mathbf{p}_2\mathbf{p}_1)=\exp\left\{-\frac{b^2}{4}\cdot\frac{a^2}{a^2+b^2}(p_1^2+p_2^2+\mathbf{p}_1\cdot\mathbf{p}_2)-\frac{3b^4}{16}\cdot\frac{1}{a^2+b^2}p_2^2\right\} \quad (10.11)$$

Calculation of λ_0 . The calculation of T_0 and Φ_0 is quite simple. We have

$$\begin{aligned} T_0 &= \int \int (p_0^2 + p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2) [\chi_0(\mathbf{p}_1\mathbf{p}_2)]^2 d\mathbf{p}_1 d\mathbf{p}_2 = \\ &= \left(p_0^2 - \frac{1}{b} \frac{\partial}{\partial b} \right) \int \int \exp \left\{ -\frac{b^2}{2} (p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2) \right\} d\mathbf{p}_1 d\mathbf{p}_2 = \\ &= \frac{(4\pi)^3}{3b^6 \sqrt{3}} \left(p_0^2 + \frac{6}{b^2} \right) \end{aligned} \quad (10.12)$$

where the integration formula (4.11) is used. Furthermore,

$$\begin{aligned} \Phi_0 &= \int \int (p_0^2 + p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2) \chi_0(\mathbf{p}_1\mathbf{p}_2) \overline{\chi_1(\mathbf{p}_1\mathbf{p}_2)} d\mathbf{p}_1 d\mathbf{p}_2 = \\ &= \frac{a}{(a^2+b^2)^{\frac{3}{2}}} \int \int \left[(1-2g) \exp \left\{ -\frac{b^2}{16} \cdot \frac{8a^2+7b^2}{a^2+b^2} (p_1^2+p_2^2) - \frac{b^2}{8} \cdot \right. \right. \\ &\quad \cdot \frac{4a^2+5b^2}{a^2+b^2} \mathbf{p}_1 \cdot \mathbf{p}_2 \left. \right\} + 2 \left(1 - \frac{g}{2} \right) \exp \left\{ -\frac{b^2}{4} \cdot \frac{2a^2+b^2}{a^2+b^2} (p_1^2 + \mathbf{p}_1 \cdot \mathbf{p}_2) - \right. \\ &\quad \left. \left. - \frac{b^2}{16} \cdot \frac{8a^2+7b^2}{a^2+b^2} p_2^2 \right\} \right] d\mathbf{p}_1 d\mathbf{p}_2 = \frac{(4\pi)^3 a}{b^6 \sqrt{3}} \cdot \frac{1-g}{\left(a^2 + \frac{b^2}{2} \right)^{\frac{3}{2}}} \end{aligned} \quad (10.13)$$

The factor $1-g$ indicates that the mean value of the potential energy derives solely from ordinary forces while the spin exchange forces have a vanishing resultant. Remembering that $\chi_0(\mathbf{p}_1\mathbf{p}_2)$ is symmetrical in all three particles, we express the latter fact simply by

$$\overline{P_{12} + P_{13} + P_{23}} = 0 \quad (10.14)$$

according to (10.8).

We now determine the minimum of the quotient

$$\lambda_0 = \frac{T_0}{\Phi_0} = \frac{1}{3a} \cdot \frac{1}{1-g} \cdot \left(p_0^2 + \frac{6}{b^2} \right) \left(a^2 + \frac{b^2}{2} \right)^{\frac{3}{2}} \quad (10.15)$$

and find, using the equations (10.3) and (10.4) and putting $b^2 = 2a^2/k$,

$$\lambda_0 = B_0 M a^2 = \frac{2}{3} \cdot \frac{1}{1-g} \cdot \frac{(k+1)^{\frac{5}{2}}}{k^{\frac{1}{2}}} \quad (10.16)$$

$$a^2 p_0^2 = -EM a^2 = k(2k-1) \quad (10.17)$$

The result has formal resemblance to that obtained for the deuteron (4.14 and 4.15) which fact facilitates the numerical calculations.

Calculation of $\lambda_{\frac{1}{2}}$. The quantity T_1 is expressed by the integral

$$T_1 = \int \int (p_0^2 + p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2) [\chi_1(\mathbf{p}_1 \mathbf{p}_2)]^2 d\mathbf{p}_1 d\mathbf{p}_2 \quad (10.18)$$

After forming the mean values of the spin operators, we find that T_1 is a sum of nine integrals all of which belong to the type

$$I(\alpha\beta\gamma) = \int \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{p_0^2 + p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2} \exp \{ -\alpha p_1^2 - \beta p_2^2 - 2\gamma \mathbf{p}_1 \cdot \mathbf{p}_2 \} \quad (10.19)$$

This integral can be computed by the method used in Section 4, the result being

$$I(\alpha\beta\gamma) = \pi^3 \int_0^\infty dt \exp \{ -p_0^2 t \} \left| \begin{array}{cc} \alpha + t & \gamma + \frac{t}{2} \\ \gamma + \frac{t}{2} & \beta + t \end{array} \right|^{-\frac{3}{2}} \quad (10.20)$$

Furthermore, several of the nine integrals are identical, so that

$$T_1 = \left(\frac{2\pi}{a^2 + b^2} \right)^3 \cdot \frac{a^2 p_0^4}{V\sqrt{3}} [(1 - 2g + 2g^2) K_1 + (2 - 4g + g^2) K_2] \quad (10.21)$$

where

$$K_1 = K \left(\frac{b^2 p_0^2}{2}, \frac{b^2 p_0^2}{2} \cdot \frac{a^2}{a^2 + b^2} \right) \quad (10.22)$$

$$K_2 = K \left(\frac{b^2 p_0^2}{8} \cdot \frac{4a^2 + 3b^2}{a^2 + b^2}, \frac{b^2 p_0^2}{8} \cdot \frac{4a^2 + b^2}{a^2 + b^2} \right) \quad (10.23)$$

$$K(\mu_1 \mu_2) = \int_0^\infty dt e^{-t} [(t + \mu_1)(t + \mu_2)]^{-\frac{3}{2}} \quad (10.24)$$

The square bracket of (10.21) can also be written

$$[] = (1-g)^2 (K_1 + 2K_2) + g^2 (K_1 - K_2) \quad (10.25)$$

Since $K_1 > K_2$, which can be proved easily, this result means that the introduction of spin forces increases the value of T_1 by the relative correction

$$\left(\frac{g}{1-g} \right)^2 \cdot \frac{K_1 - K_2}{K_1 + 2K_2} \quad (10.26)$$

giving approximately the same relative decrease of $\lambda_{\frac{1}{2}}$. As before we put $b^2 = 2a^2/k$ and find

$$\lambda_{\frac{1}{2}} = B_2 Ma^2 = \frac{(k+2)^3}{(2k-1)^2 k^{\frac{1}{2}} (k+1)^{\frac{3}{2}}} \cdot \frac{1-g}{(1-g)^2 (K_1 + 2K_2) + g^2 (K_1 - K_2)} \quad (10.27)$$

$$K_1 = K \left(2k-1, (2k-1) \cdot \frac{k}{k+2} \right) \quad (10.28)$$

$$K_2 = K \left((2k-1) \cdot \frac{k+\frac{3}{2}}{k+2}, (2k-1) \cdot \frac{k+\frac{1}{2}}{k+2} \right) \quad (10.29)$$

where the function K is that defined above (Eq. 10.24).

The integral $K(\mu_1 \mu_2)$ is evaluated by means of a series expansion of $[(t+\mu_1)(t+\mu_2)]^{-\frac{3}{2}}$ with respect to

$$\frac{\mu_1 - \mu_2}{t + \frac{1}{2}(\mu_1 + \mu_2)}$$

and integration term-by-term.

We find

$$K(\mu_1 \mu_2) = \left(\frac{2}{\mu_1 + \mu_2} \right)^3 \sum_{r=0}^{\infty} \binom{-\frac{3}{2}}{r} (-1)^r \left(\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \right)^{2r} G_{2r+3} \left(\frac{\mu_1 + \mu_2}{2} \right) \quad (10.30)$$

where the functions

$$G_n(x) = \int_0^{\infty} dt e^{-t} \left(\frac{x}{x+t} \right)^n \quad (10.31)$$

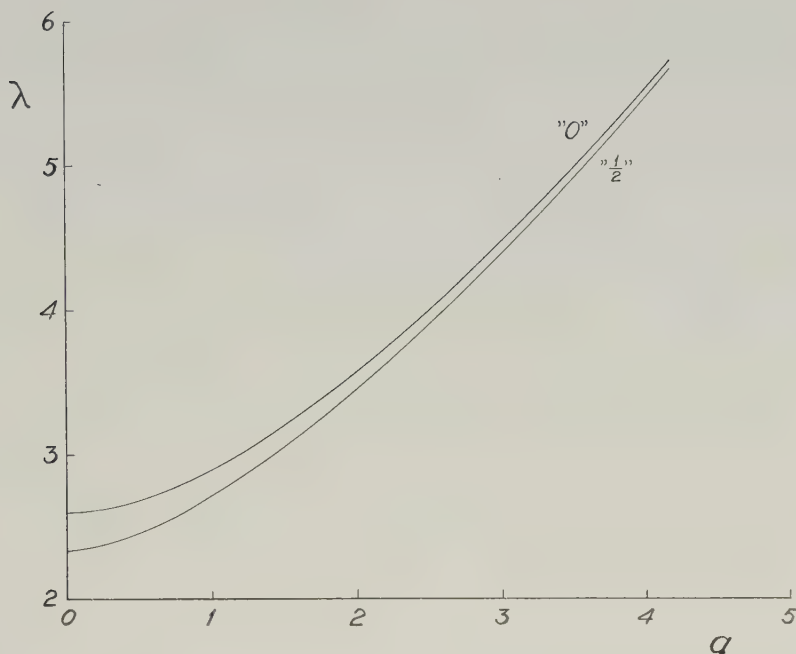


Figure 2. The force constant for the triton as a function of the range a (in 10^{-13} cm).

are computed by means of the recurrence formula

$$G_{n+1}(x) = \frac{x}{n} [1 - G_n(x)] \quad (10.32)$$

while

$$G_1(x) = \int_x^\infty dt e^{-t} \frac{x}{t} = -x e^x Ei(-x) \quad (10.33)$$

where $Ei(x)$ is the exponential integral. (See B. A. Mathem. Tables, Vol. I.)

The curves »O» and » $\frac{1}{2}$ » of Figure 2 represent λ_0 and $\lambda_{\frac{1}{2}}$ respectively with $g = 0$. The relative correction $(\lambda_0 - \lambda_{\frac{1}{2}}) / \lambda_0$ is less than in the deuteron case. This fact presumably entitles us to regard the pure variation method as being more favourable in the case of the triton than in that of the deuteron.

Calculation of λ_1 . For the sake of simplicity we put $g=0$. The more general case $g \neq 0$ is dealt with in the same way as above after evaluation of the mean values

$$\overline{P_{jj'} P_{kk'} P_{ll'}}$$

which is a simple task.

The quantity Φ_1 is the sum of three quantities

$$\Phi_1 = \Phi_1^{(12)} + \Phi_1^{(13)} + \Phi_1^{(23)} \quad (10.34)$$

corresponding to the three interactions. Each quantity is a sum of nine integrals of the form

$$K(\alpha\beta\gamma\delta\epsilon\zeta) = \iiint \frac{d\mathbf{p}_1 d\mathbf{p}'_1 d\mathbf{p}_2}{(p_0^2 + p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2)(p_0^2 + p_1'^2 + p_2^2 + \mathbf{p}'_1 \cdot \mathbf{p}_2)} \cdot \\ \cdot \exp \{ -\alpha p_1'^2 - \beta p_1^2 - \gamma p_2^2 - 2\delta \mathbf{p}'_1 \cdot \mathbf{p}_1 - 2\epsilon \mathbf{p}'_1 \cdot \mathbf{p}_2 - 2\zeta \mathbf{p}_1 \cdot \mathbf{p}_2 \} \quad (10.35)$$

Among these 27 integrals there are only four which are different. We have, for example,

$$\begin{aligned} \Phi_1^{(12)} &= \Phi_1^{(13)} = \Phi_1^{(23)} \\ \Phi_1^{(13)} (12 | 12) &= \Phi_1^{(13)} (23 | 23) \\ \Phi_1^{(13)} (12 | 13) &= \Phi_1^{(13)} (13 | 23) \end{aligned} \quad (10.36)$$

where the new indices refer to the indices of the functions ω_{jk} in (10.9). The quantities are, of course, symmetrical with respect to these pairs of indices. By our ordinary methods the integral (10.35) can be reduced to a double integral

$$K(\alpha\beta\gamma\delta\epsilon\zeta) = \pi^{\frac{9}{2}} \int_0^\infty ds \int_0^\infty dt \exp \{ -p_0^2(s+t) \} [\Delta(s, t)]^{-\frac{9}{2}} \quad (10.37)$$

where

$$\Delta(s, t) = \begin{vmatrix} \alpha+t & \delta & \epsilon+\frac{t}{2} \\ \delta & \beta+s & \zeta+\frac{s}{2} \\ \epsilon+\frac{t}{2} & \zeta+\frac{s}{2} & \gamma+s+t \end{vmatrix} \quad (10.38)$$

After introducing the new variables

$$s+t = 2u, \quad s-t = v$$

integration with respect to v can be performed. As the result is somewhat complicated in the general case, though expressed by elementary functions, we reproduce only the simplest case, namely for $\Phi_1^{(13)}(13 | 13)$, which has the form

$$L(\mu_1 \mu_2 \mu_3 \mu_4) = \int_0^\infty dx \frac{x e^{-x}}{(x + \mu_1)^{\frac{3}{2}} (x + \mu_2)^{\frac{1}{2}} (x + \mu_3) (x + \mu_4)} \quad (10.39)$$

where

$$\mu_1 = \frac{a^2 p_0^2}{k}, \mu_2 = \mu_1 \cdot \frac{k(k+3)}{(k+2)(k+4)}, \mu_3 = \mu_1 \cdot \frac{k}{k+2}, \mu_4 = \mu_1 \cdot \frac{k(k+3)}{k+2}$$

This integral must be computed by numerical integration. It is convenient to transform the integral into a finite interval of integration and then apply the Simpson formula.

The numerical calculations have been performed for $k=1$, which gives $a^2 p_0^2 = 1$ according to (10.17). Assuming that $E = -8.38$ Mev (FLÜGGE-MATTAUCH 1941), this corresponds to $a = 2.22 \cdot 10^{-13}$ cm for the range of the forces. This value is close to those assumed by other authors: *e. g.* $a = 2.25 \cdot 10^{-13}$ cm by MARGENAU and WARREN (1937).

Still assuming $g=0$, we obtain as a result of our calculation

$$\underline{\lambda_1 = 3.630}$$

The corresponding figures for λ_0 and $\lambda_{\frac{1}{2}}$ (in the case $k=1$) are

$$\lambda_0 = 3.771 \text{ and } \lambda_{\frac{1}{2}} = 3.662$$

It has already been pointed out that the difference $\lambda_0 - \lambda_{\frac{1}{2}}$ is somewhat smaller in the case of the triton than in that of the deuteron. On the other hand the difference $\lambda_{\frac{1}{2}} - \lambda_1$ is about three times as large in the triton case. One reason for this state of affairs is that the radius of the nucleus, expressed by $1/p_0$, is just equal to the range of the forces: for larger values of a the pure variational method (λ_0) will be a better approximation and for smaller values of a the improvement achieved by the iteration ($\lambda_{\frac{1}{2}}$) will be more effective.

11. The Coulomb Energy of He^3 .

If the potential energy of two nucleons is independent of their isotopic states, as we have assumed for the triton, the stationary states of He^3 will differ from those of H^3 only by an energy amount arising from the repulsive Coulomb force between the two protons in He^3 .

The eigenvalue problem can be formulated as before by an integral equation in which the Coulomb repulsion appears as a term

$$\frac{e^2}{2\pi^2} \int \frac{1}{(\mathbf{p}_1 - \mathbf{p}'_1)^2} \varphi(\mathbf{p}'_1, \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1) d\mathbf{p}'_1 \quad (11.1)$$

in which the indices 1 and 2 refer to the protons. Since the effect of this term is, however, quite small, the ordinary perturbation theory can be applied, giving the energy correction E_c as a mean value

$$E_c = \overline{\left(\frac{e^2}{r} \right)} \quad (11.2)$$

of e^2/r with the eigenfunctions of the triton used as unperturbed functions. It implies a further simplification to take the approximate eigenfunction (10.7) for the triton, whereby we obtain

$$E_c^{(0)} = \frac{e^2}{2\pi^2 N} \int \int \int \frac{1}{(\mathbf{p}_1 - \mathbf{p}'_1)^2} \exp \left\{ -\frac{b^2}{4} (p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2 + p_1'^2 + p_2'^2 + \mathbf{p}_1' \cdot \mathbf{p}_2) \right\} d\mathbf{p}'_1 d\mathbf{p}_1 d\mathbf{p}_2 \quad (11.3)$$

where

$$N = \int \int \exp \left\{ -\frac{b^2}{2} (p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2) \right\} d\mathbf{p}_1 d\mathbf{p}_2 \quad (11.4)$$

By means of our ordinary integration methods we find

$$E_c^{(0)} = \frac{e^2}{b} \sqrt{\frac{8}{\pi}} = \frac{2e^2}{a\sqrt{\pi}} \cdot V\bar{k} \quad (11.5)$$

If the iterated function $\chi_1(\mathbf{p}_1 \mathbf{p}_2)$ (Equation 10.9) is used, the result may be written

$$E_c^{(1)} = \frac{2e^2}{a\sqrt{\pi}} \cdot \sqrt{k} \cdot m_c \quad (11.6)$$

where the factor $m_c = C/N$ expresses the improvement caused by the use of the iterated function. The quantity C as well as the normalization factor N is a sum of nine integrals corresponding to the nine combinations of the functions ω_{jk} . The integrals in the factor N are of only two different types, three diagonal terms

$$N(12 | 12) = N(23 | 23) = N(31 | 31)$$

and six non-diagonal

$$N(12 | 13) = N(12 | 23) = N(13 | 23) = \dots$$

being equal. These two groups of integrals correspond to the two integrals K_1 and K_2 appearing in (10.21) and are computed in a similar way. Among the integrals of C four are different

$$\begin{array}{ll} C(12 | 12) & 1 \\ C(13 | 13) = C(23 | 23) & 2 \\ C(12 | 13) = C(12 | 23) = \dots & 4 \\ C(13 | 23) = C(23 | 13) & 2 \\ \hline & 9 \end{array}$$

The first of these is the simplest but is typical:

$$\begin{aligned} C(12 | 12) = & \int \int \int \frac{\omega_{12}(\mathbf{p}_1 \mathbf{p}_2)}{T(\mathbf{p}_1 \mathbf{p}_2)} \cdot \frac{1}{(\mathbf{p}_1 - \mathbf{p}'_1)^2} \cdot \\ & \cdot \frac{\omega_{12}(\mathbf{p}'_1, \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1)}{T(\mathbf{p}'_1, \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1)} d\mathbf{p}'_1 d\mathbf{p}_1 d\mathbf{p}_2 \end{aligned} \quad (11.7)$$

where

$$T(\mathbf{p}_1 \mathbf{p}_2) = p_0^2 + p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2 \quad (11.8)$$

As before we find

$$C(12 | 12) = \pi^{\frac{3}{2}} \int_0^\infty ds \int_0^\infty dt \int_0^\infty du \exp \{ -p_0^2 (s+t) \} A^{-\frac{3}{2}} \quad (11.9)$$

where

$$\Delta = \left[\beta_2 + \frac{3}{2}(s+t) \right] \left[\frac{1}{2}(\beta_1 + s)(\beta_1 + t) + \beta_1 u + \frac{1}{2}(s+t)u \right] \quad (11.10)$$

$$\beta_1 = \frac{b^2}{4} \cdot \frac{a^2}{a^2 + b^2}, \quad \beta_2 = \frac{3b^2}{4}$$

Here two integrations can be performed by elementary means, namely with respect to u and $s - t$. The result is

$$C(12|12) = \frac{16\pi^{\frac{3}{2}}}{\sqrt{2}} \int_0^\infty dx \exp\{-2p_0^2 x\} (\beta_1 + x)^{-1} (\beta_2 + 3x)^{-\frac{3}{2}} \cdot \arcsin \frac{x}{x + \beta_1} \quad (11.11)$$

This integral is computed numerically.

The result of the calculations shows a very small improvement. We obtain

$$\begin{array}{lll} \text{for } k = 1 & \text{and } g = 0 & m_c = 0.998 \\ \text{for } k = 1 & \text{and } g = 0.2 & m_c = 0.972 \end{array}$$

This fact is quite understandable, since the Coulomb energy e^2/r receives small contributions from those regions of the coordinate space where the wave function $\psi_0(\mathbf{r}_1 \mathbf{r}_2 \mathbf{r}_3)$ is asymptotically wrong. We must also remember that the initial function $\chi_0(\mathbf{p}_1 \mathbf{p}_2)$ is chosen symmetrical with respect to an exchange of the coordinates of the nucleons. At least in the case when g is large, this is a poor approximation, which is possibly not remedied sufficiently in the iterated function $\chi_1(\mathbf{p}_1 \mathbf{p}_2)$. If an asymmetrical wave function

$$\chi_0(\mathbf{p}_1 \mathbf{p}_2) = \exp \left\{ -\frac{b_1^2}{4}(p_1^2 + p_2^2) - \frac{b_2^2}{4} \mathbf{p}_1 \cdot \mathbf{p}_2 \right\} \quad (11.12)$$

is introduced from the very beginning and the parameters b_1 and b_2 are determined so that λ_0 is a minimum, we obtain an expression for the Coulomb energy analogous to Equation (11.5)

$$E_c^{(0)} = \frac{2e^2}{a\sqrt{\pi}} \cdot \sqrt{k} \cdot m_c \quad (11.13)$$

The correctional factor m_c due to the asymmetry is

$$m_c = 0.935 \quad \text{for } g = 0.2.$$

Details of the calculation of λ_0 are reproduced in Section 13 (*cf.* BB § 22). It may happen that the value of m_e is somewhat too low, inasmuch as λ_0 is relatively insensitive to the degree of asymmetry. On the other hand the Coulomb interaction in itself promotes the asymmetry. In any case the accuracy of our results is sufficient for comparison with the experimentally known difference between the binding energies of H^3 and He^3 (MATTAUCH-FLÜGGE, Table VI).

12. The Ground State of the Alpha Particle.

With the same assumptions regarding the nuclear forces as were made in the triton case, the integral equation of the α -particle in the ground state can be written

$$\begin{aligned} & (p_0^2 + p_1^2 + p_2^2 + p_3^2 + \mathbf{p}_1 \cdot \mathbf{p}_2 + \mathbf{p}_1 \cdot \mathbf{p}_3 + \mathbf{p}_2 \cdot \mathbf{p}_3) \varphi(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3 | 1234) = \\ & = \lambda a (4\pi)^{-\frac{3}{2}} \left[\sum_{123} \int (1 - g + g P_{jk}) \exp \left\{ -\frac{a^2}{4} (\mathbf{p}_j - \mathbf{p}'_j)^2 \right\} \cdot \right. \\ & \quad \cdot \varphi(\mathbf{p}'_j, \mathbf{p}_j + \mathbf{p}_k - \mathbf{p}'_j, \mathbf{p}_l | \widetilde{12} \widetilde{34}) d\mathbf{p}'_j + \\ & \quad \left. + \sum_{123} \int (1 - g + g P_{jl}) \exp \left\{ -\frac{a^2}{4} (\mathbf{p}_j - \mathbf{p}'_j)^2 \right\} \varphi(\mathbf{p}'_j, \mathbf{p}_k, \mathbf{p}_l | \widetilde{12} \widetilde{34}) d\mathbf{p}'_j \right] \end{aligned} \quad (12.1)$$

The neutrons are denoted by 1 and 2, the protons by 3 and 4. The summations are extended over the cyclic permutations of the indices $jk l$. In the right-hand member of Equation (12.1) the wave function is assumed to be symmetrical in the position coordinates and antisymmetrical in the spin coordinates of pairs of like particles, thus anticipating the true properties of the wave function in the case $g = 0$ and the approximate properties for $g \neq 0$. The Coulomb interaction energy will be calculated separately by the perturbation method.

For the spin operators P_{jk} the following mean values are needed:

$$\begin{aligned} \overline{P_{12}} = \overline{P_{34}} = -1, \quad \overline{P_{13}} = \overline{P_{14}} = \overline{P_{23}} = \overline{P_{24}} = \frac{1}{2} \\ \overline{P_{12} P_{34}} = \overline{P_{13} P_{24}} = \overline{P_{14} P_{23}} = 1 \\ \overline{P_{jk} P_{jl}} = -\frac{1}{2}, \quad j \neq k \neq l \end{aligned} \quad (12.2)$$

Our initial function

$$\chi_0(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3) = \exp \left\{ -\frac{b^2}{4} (p_1^2 + p_2^2 + p_3^2 + \mathbf{p}_1 \cdot \mathbf{p}_2 + \mathbf{p}_1 \cdot \mathbf{p}_3 + \mathbf{p}_2 \cdot \mathbf{p}_3) \right\} \quad (12.3)$$

is the Fourier transform of a well-known trial function in the theory of the α -particle. The iterated function is written explicitly

$$\begin{aligned} T(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3) \chi_1(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3) = & \frac{a}{(a^2 + b^2)^{\frac{3}{2}}} \left[\sum_{123} (1 - g + g P_{jk}) \exp \left\{ -\frac{b^2}{16} \cdot \right. \right. \\ & \cdot \frac{4a^2 + 3b^2}{a^2 + b^2} (p_j^2 + p_k^2) - \frac{b^2}{8} \cdot \frac{2a^2 + 3b^2}{a^2 + b^2} \cdot \mathbf{p}_j \cdot \mathbf{p}_k - \frac{b^2}{4} (\mathbf{p}_j \cdot \mathbf{p}_l + \mathbf{p}_k \cdot \mathbf{p}_l + p_l^2) \left. \right\} + \\ & \cdot \sum_{123} (1 - g + g P_{jl}) \cdot \exp \left\{ -\frac{b^2}{16} \cdot \frac{4a^2 + 3b^2}{a^2 + b^2} (p_k^2 + p_l^2) - \frac{b^2}{8} \cdot \frac{2a^2 + b^2}{a^2 + b^2} \cdot \right. \\ & \left. \left. \mathbf{p}_k \cdot \mathbf{p}_l - \frac{b^2}{4} \cdot \frac{a^2}{a^2 + b^2} (p_j^2 + \mathbf{p}_j \cdot \mathbf{p}_k + \mathbf{p}_j \cdot \mathbf{p}_l) \right\} \right] \quad (12.4) \end{aligned}$$

$$T(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3) = p_0^2 + p_1^2 + p_2^2 + p_3^2 + \mathbf{p}_1 \cdot \mathbf{p}_2 + \mathbf{p}_1 \cdot \mathbf{p}_3 + \mathbf{p}_2 \cdot \mathbf{p}_3 \quad (12.5)$$

Calculation of λ_0 . As in the case of the triton we calculate

$$T_0 = \iiint T(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3) [\chi_0(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3)]^2 d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 = \frac{2^6 \pi^{\frac{9}{2}}}{b^9} \left(p_0^2 + \frac{9}{b^2} \right) \quad (12.6)$$

$$\begin{aligned} \Phi_0 = & \iiint T(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3) \overline{\chi_0(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3)} \chi_1(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3) d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3 = \\ & = \frac{3 \cdot 2^7 \cdot \pi^{\frac{9}{2}} \cdot a}{b^9} \cdot \frac{1 - g}{\left(a^2 + \frac{b^2}{2} \right)^{\frac{3}{2}}} \quad (12.7) \end{aligned}$$

The minimum value of $\lambda_0 = T_0 / \Phi_0$ with respect to b is

$$\lambda_0 = B_0 M a^2 = \frac{1}{2} \cdot \frac{1}{1 - g} \cdot \frac{(k + 1)^{\frac{5}{2}}}{k^{\frac{1}{2}}} \quad (12.8)$$

$$a^2 p_0^2 = -E M a^2 = 3k \left(k - \frac{1}{2} \right) \quad (12.9)$$

Calculation of $\lambda_{\frac{1}{2}}$. Since $\chi_1(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3)$ is the sum of six different terms corresponding to the six pairs of interactions, the quantity T_1

will be the sum of 36 integrals. There are, however, only three different integrals, all of the type

$$K(\mu_1 \mu_2 \mu_3) = \int_0^\infty dt e^{-t} [(t + \mu_1)(t + \mu_2)(t + \mu_3)]^{-\frac{3}{2}} \quad (12.10)$$

The calculations are for the rest analogous to those in the triton case and give as a result

$$\lambda_{\frac{1}{2}}^2 = \frac{1}{\left[3 \left(k - \frac{1}{2}\right)\right]^{\frac{1}{2}}} \cdot \frac{(k+2)^3}{k^{\frac{1}{2}} (k+1)^{\frac{3}{2}}} \cdot \frac{1-g}{(1-g)^2 (K_1 + K_2 + K_3) + g^2 (K_1 - K_2 + K_3)} \quad (12.11)$$

where

$$K_1 = K\left(\mu, \mu, \mu \cdot \frac{k}{k+2}\right) \quad (12.12)$$

$$K_2 = K\left(\mu, \mu \cdot \frac{k+\frac{1}{2}}{k+2}, \mu \cdot \frac{k+\frac{3}{2}}{k+2}\right) \quad (12.13)$$

$$K_3 = K\left(\mu, \mu \cdot \frac{k+1}{k+2}, \mu \cdot \frac{k+1}{k+2}\right) \quad (12.14)$$

$$\mu = 3 \left(k - \frac{1}{2}\right)$$

The type integral $K(\mu_1 \mu_2 \mu_3)$ has been computed by series expansion of the factor $[(t + \mu_1)(t + \mu_2)(t + \mu_3)]^{-\frac{3}{2}}$ into negative powers of $(t + \bar{\mu})$ where $\bar{\mu} = \frac{1}{3}(\mu_1 + \mu_2 + \mu_3)$ and integration term-by-term.

The quantity $\bar{\mu}$ has the same value in the three cases K_1, K_2, K_3 ,

namely $\bar{\mu} = \mu \cdot \frac{k+\frac{4}{3}}{k+2}$. We obtain in this way

$$K(\mu_1 \mu_2 \mu_3) = (\bar{\mu})^{-\frac{3}{2}} \sum_{\nu=0}^{\infty} R_{\nu}(a, \beta) G_{\nu+\frac{3}{2}}(\bar{\mu}) \quad (12.15)$$

The functions $G_{n+\frac{1}{2}}$ have already been used in Section 6 (Appendix I, Formulae 8 and 9). The coefficients $R_{\nu}(a, \beta)$ are polynomials

in the parameters α and β and are defined as the coefficients of τ^v in the expansion

$$(1 - 3\alpha\tau^2 - \beta\tau^3)^{-\frac{3}{2}} = \sum_{v=0}^{\infty} R_v(\alpha, \beta) \tau^v \quad (12.17)$$

They satisfy the recurrence formula

$$(\nu+1)R_{\nu+1}(\alpha, \beta) = 3\alpha(\nu+2)R_{\nu-1}(\alpha, \beta) + \beta\left(\nu + \frac{5}{2}\right)R_{\nu-2}(\alpha, \beta) \quad (12.18)$$

$$R_0(\alpha, \beta) = 1, \quad R_1(\alpha, \beta) = 0$$

The parameters α and β have the values

$$\alpha = \alpha' \cdot \frac{\mu^2}{9(k+2)^2}, \quad \beta = \beta' \cdot \frac{\mu^3}{27(k+2)^3} \quad (12.19)$$

where α' and β' are numerical coefficients which have the following values in the three cases K_1 , K_2 , K_3 :

	K_1	K_2	K_3
$\alpha' =$	4	$\frac{7}{4}$	1
$\beta' =$	16	$\frac{5}{2}$	-2

The curves »O» and » $\frac{1}{2}$ » of Figure 3 represent λ_0 and $\lambda_{\frac{1}{2}}$ as functions of the parameter k in the case $g=0$. The relative correction $(\lambda_0 - \lambda_{\frac{1}{2}})/\lambda_0$ is now even smaller than in the triton case.

13. The Rôle of Exchange Forces in the Three- and Four-body Problems.

By the term *exchange forces* we mean those types of forces which were introduced in nuclear theory by HEISENBERG and MAJORANA in order to explain the saturation of energy in nuclei heavier than the α -particle and which are represented by a potential energy term including an operator which exchanges the position coordinates of two interacting nucleons. If the spin coordinates are also exchanged, the force is called a pure Heisenberg force, while if only the position coordinates are exchanged, the force is of the pure Majorana type. In earlier investigations concerning the binding energies of the triton and the α -particle, the interactions have frequently been assumed to be entirely of the exchange type,

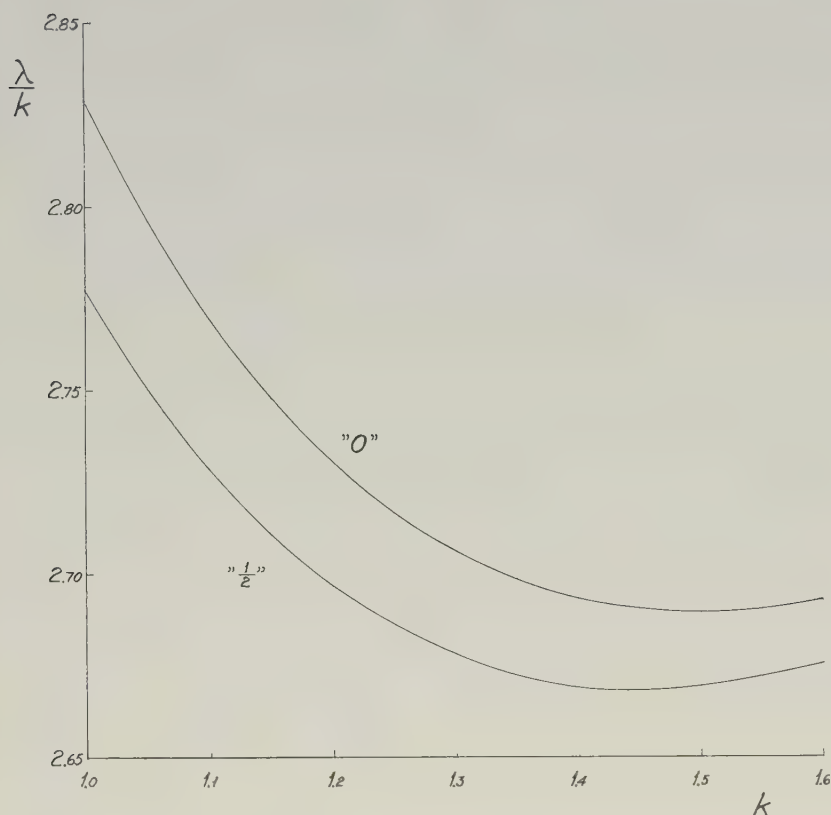


Figure 3. The ratio λ/k for the α -particle as a function of the parameter k .

but some authors (*e.g.* BREIT and FEENBERG 1936) have pointed out that an arbitrarily large admixture of ordinary (and spin exchange) forces could be admitted without changing the numerical results in any noteworthy manner. The reason for this is that the wave function for the ground state is nearly symmetrical in the position coordinates even for a comparatively large value of the constant g . There are two different ways of proceeding to estimate the effect of exchange forces in our scheme.

We might account for the asymmetry of the wave function by putting

$$\chi_0(\mathbf{p}_1 \mathbf{p}_2) = \exp \left\{ -\frac{3b_1^2}{16} (\mathbf{p}_1 + \mathbf{p}_2)^2 - \frac{b_2^2}{16} (\mathbf{p}_1 - \mathbf{p}_2)^2 \right\} \quad (13.1)$$

for the triton and minimizing λ_0 with respect to the parameters b_1 and b_2 . We might also add a term containing one of the spin operators P_{13} or P_{23} , which would imply that the neutrons were partly in a symmetrical spin state. This method would be unnecessarily complicated, since at least three variational parameters would be needed.

On the other hand, in applying the V—I method we are not prevented from retaining the simple initial function $\chi_0(\mathbf{p}_1\mathbf{p}_2)$, Eq. (10.7), and allowing the effect of the exchange operators to appear in the iterated functions. In this way we find that the quantities T_0 , Φ_0 and T_1 are identical for ordinary and exchange¹ forces, and that a splitting effect first enters the quantity Φ_1 , that is λ_1 . Since the calculation of λ_1 is very complicated in such a case, though possible, we prefer the first method and discuss the splitting effect only in so far as it concerns λ_0 .

With the wave function (13.1) we calculate for ordinary (including spin exchange) forces the following quantities

$$T_0 = \left(\frac{4\pi}{b_1 b_2 \sqrt{3}} \right)^3 \left(p_0^2 + \frac{3}{b_1^2} + \frac{3}{b_2^2} \right) \quad (13.2)$$

$$\Phi_0 = (1-2g) \Phi_0^{(12)} + \left(1 - \frac{1}{2}g \right) (\Phi_0^{(13)} + \Phi_0^{(23)}) \quad (13.3)$$

$$\Phi_0^{(12)} = \left(\frac{4\pi}{b_1 b_2 \sqrt{3}} \right)^3 \cdot a \cdot \left(a^2 + \frac{1}{2} b_2^2 \right)^{-\frac{3}{2}} \quad (13.4)$$

$$\Phi_0^{(13)} = \Phi_0^{(23)} = \left(\frac{4\pi}{b_1 b_2 \sqrt{3}} \right)^3 \cdot a \cdot \left(a^2 + \frac{3}{8} b_1^2 + \frac{1}{8} b_2^2 \right)^{-\frac{3}{2}} \quad (13.5)$$

The minimum value of $\lambda_0 = T_0 / \Phi_0$ with respect to b_1 and b_2 is

$$\lambda_0 = \frac{2}{3} \cdot \frac{1}{1 - \frac{1}{2}g} \cdot k_1^{-\frac{1}{2}} \left(k_1 + \frac{3}{4} + \frac{k_1}{4k_2} \right)^{\frac{5}{2}} \quad (13.6)$$

with the conditions

$$a^2 p_0^2 = k_1^2 + k_2^2 - \frac{1}{2}(k_1 + k_2) \quad (13.7)$$

$$\left(k_1 - \frac{3}{4} \cdot \frac{k_1 - k_2}{k_2 + 1} \right)^{\frac{5}{2}} = \frac{1 - \frac{1}{2}g}{1 - 2g} \cdot \frac{k_1^{\frac{1}{2}}}{2} (3k_2^2 - k_1^2) \quad (13.8)$$

¹ It may be observed that the expression corresponding to the potential energy is not positive-definite in this case.

With a representative choice of the parameters, namely

$$a^2 p_0^2 = 1.0142, \quad g = 0.2$$

we find

$$\lambda_0 = 4.712$$

which value is to be compared with

$$\lambda_0 = 4.730$$

obtained with a symmetrical wave function.

The difference between the corresponding values of the force constant is about 0.15 Mev for a range of $a = 2.25 \cdot 10^{-13}$ cm.

If the forces are of the Majorana and Heisenberg types, we should apply exchange operators Q_{12} , Q_{13} , Q_{23} to the three integrals of the right-hand member of Equation (10.2) respectively. These operators act upon the momentum coordinates in the following way:

$$\begin{aligned} Q_{12} \{ \mathbf{p}_1, \mathbf{p}_2 \} &= \{ \mathbf{p}_2, \mathbf{p}_1 \} \\ Q_{13} \{ \mathbf{p}_1, \mathbf{p}_2 \} &= \{ -\mathbf{p}_1 - \mathbf{p}_2, \mathbf{p}_2 \} \\ Q_{23} \{ \mathbf{p}_1, \mathbf{p}_2 \} &= \{ \mathbf{p}_1, -\mathbf{p}_1 - \mathbf{p}_2 \} \end{aligned} \quad (13.9)$$

Only the quantities $\Phi_0^{(13)}$ and $\Phi_0^{(23)}$ of our last calculation are influenced by these operators.

We find

$$\Phi_0^{(13)} = \Phi_0^{(23)} = \left(\frac{4\pi}{b_1 b_2} \right)^3 \cdot a \cdot \left[a^2 + \frac{3}{8} b_1^2 + \frac{1}{8} b_2^2 + \frac{3}{16} a^2 \left(\frac{b_1}{b_2} - \frac{b_2}{b_1} \right)^2 \right]^{-\frac{3}{2}} \quad (13.10)$$

As the difference $b_2^2 - b_1^2$ is quite small in comparison with $b_1 b_2$, the relative change of λ_0 due to the exchange forces may be approximately written

$$\begin{aligned} \frac{\Delta \lambda_0}{\lambda_0} &= \frac{3}{16} \cdot \frac{1 - \frac{1}{2}g}{1 - g} \cdot \frac{a^2}{b_1^2 b_2^2} \cdot (b_1^2 - b_2^2)^2 \left(a^2 + \frac{3}{8} b_1^2 + \frac{1}{8} b_2^2 \right)^{-1} = \\ &= \frac{3}{4} \cdot \frac{1 - \frac{1}{2}g}{1 - g} \cdot \frac{(k_1 - k_2)^2}{k_1 + 3k_2 + 4k_1 k_2} \end{aligned} \quad (13.11)$$

With the same values of the constants g , k_1 , k_2 as above we obtain as an example

$$\frac{\Delta\lambda_0}{\lambda_0} = 0.00307$$

which corresponds to an increase

$$\Delta B_0 = 0.12 \text{ Mev}$$

in the value of B_0 .

With an analogous calculation for the α -particle we obtain in an actual case

$$\frac{\Delta\lambda_0}{\lambda_0} = 0.00155$$

This figure is about one half of the value obtained for the triton, which is quite plausible.

The wave function employed for the α -particle in this case is

$$\chi_0(\mathbf{p}_1 \mathbf{p}_2 \mathbf{p}_3) = \exp \left\{ -\frac{b_1^2}{4} [(\mathbf{p}_1 + \mathbf{p}_3)^2 + (\mathbf{p}_2 + \mathbf{p}_3)^2] - \frac{b_2^2}{4} (\mathbf{p}_1 + \mathbf{p}_2)^2 \right\} \quad (13.12)$$

which is symmetrical in like particles as well as in the two pairs of like particles. This choice of symmetry is correct if the Coulomb interaction is left out of consideration.

14. Discussion of the Results. Comparison with Earlier Investigations.

Considerable efforts have been made by several authors to construct a unique law of force which accounts consistently for the binding energies of the lightest nuclei H^2 , H^3 , He^3 and He^4 . With the same end, increasing attention has been devoted to scattering experiments, in particular those concerning the scattering of neutrons and protons by protons. There is at least one reason for pursuing the discussion of the force problem based on these two groups of experimental data. The exchange force hypothesis, so essential for the theory of heavy nuclei, is almost irrelevant to

the calculation of the binding energies of H^3 and He^4 , as we have seen in the last section, as well as to the interpretation of the scattering data, except for high energies. Furthermore, the ground state of the deuteron is in no way influenced by the exchange hypothesis. With respect to the recent experiments on the scattering of high-energy neutrons by protons (*cf.* Introduction), it is quite advantageous that the latter hypothesis can be kept out of the discussion for the present.

During the last years the deuteron problem has enjoyed a prominent position. For the numerous variants of meson theories it has served as a touchstone. In this connexion we direct our attention upon the non-central forces, mentioned in the introduction to Chapter II. If these forces are responsible for some part of the binding energy of the deuteron, they play a minor part in the binding energies of H^3 and He^4 . An obvious reason for this is that the ground state of H^2 is a pure triplet state irrespective of the amount of admixture of non-central forces, while the ground states of the nuclei H^3 and He^4 are a doublet and a singlet state respectively, in the absence of non-central forces. These states, which belong to the lowest possible values of the total spin quantum number, are coupled by the non-central forces with states of higher multiplicity. An attempt to introduce non-central forces into the three- and four-body problems has been made by GERJUOY and SCHWINGER (1942).

An interaction operator, by means of which the properties of the deuteron are represented (RARITA and SCHWINGER 1941) and in which the non-central force dominates over other kind of forces, cannot account for the binding energies of H^3 and He^4 , according to Gerjuoy and Schwinger. Though their calculations are somewhat incomplete, it may be observed that the computed values of the binding energies are much too small, showing that non-central forces are rather ineffective in binding H^3 and He^4 . On these grounds it seems justified to exclude the deuteron problem from our present discussion. Thus we may compare the results of the calculations in the Sections 10, 11, 12, 13 with the results of earlier investigations by other authors and with the results from the analysis of scattering experiments.

Some of the most accurate calculations of the energy states of the lightest nuclei have been made by Margenau and collaborators.

These authors use the error function potential throughout with the following values of constants for the triton:

$$B = 35.60 \text{ Mev}, \quad a = 2.25 \cdot 10^{-13} \text{ cm}, \quad g = 0.20$$

and with a slightly different value of g for the α -particle. The method employed is chiefly the variational method. By means of a linear set of 13 functions, MARGENAU and WARREN (1937) find for the triton an energy of -7.21 Mev, representing an upper limit of the normal state. Guided by experience from a similar calculation for the deuteron these authors estimate the convergence limit to lie at -7.7 Mev. In a later publication (MARGENAU and TYRRELL 1938) this figure is changed to -7.3 Mev as a more probable value. Bearing in mind our inversion of the eigenvalue problem, we present a comparison in the following way. With an assumed energy value $E = -7.21$ Mev we calculate by steps:

$$B_0 = 37.07 \text{ Mev}, \quad B_{\frac{1}{2}} = 35.77 \text{ Mev}, \quad B_1 = 35.38 \text{ Mev}.$$

The last value has been computed by an extrapolation from the value of B_1 (Section 10) found at the parameter value $k=1$, assuming the slope of the curve $B_1(k)$ and that of $B_{\frac{1}{2}}(k)$ to be equal in the vicinity of $k=1$.

If the value $E = -7.30$ Mev is accepted for the binding energy, we obtain in the same way

$$B_0 = 37.18 \text{ Mev}, \quad B_{\frac{1}{2}} = 35.88 \text{ Mev}, \quad B_1 = 35.49 \text{ Mev}.$$

Only one correction must be added to these values. Margenau and Warren have employed exchange forces of the Majorana and Heisenberg types; we have assumed ordinary and spin exchange forces. With the estimation of this difference made in the last section we obtain finally

$$B_1 = 35.49 \text{ Mev for } E = -7.21 \text{ Mev}$$

$$B_1 = 35.60 \text{ Mev for } E = -7.30 \text{ Mev}$$

which gives excellent support to the judgement of Margenau and Tyrrell.

With the nuclear constants adopted by Margenau and Warren the Coulomb energy of He^3 is, according to formula (11.5),

$$E_c = 0.71 \text{ Mev}$$

and with an asymmetrical wave function

$$E_c = 0.68 \text{ Mev}$$

according to Equation (11.13). These values lie within the limits of the experimentally known energy difference $E(\text{He}^3) - E(\text{H}^3)$, which are 0.66 and 1.14 Mev (MATTAUCH-FLÜGGE, 1942, Table VI).

The normal state of the α -particle has been calculated by MARGENAU and TYRRELL (1938). With an aggregate of 11 variation functions, including all doubly and quadruply excited functions in their notation, they find $E = -26.76$ Mev as an upper limit for the normal state with the following values of the force constants:

$$B = 35.60 \text{ Mev}, a = 2.25 \cdot 10^{-13} \text{ cm}, g = 0.194.$$

By an estimate of the effect of the higher (sextuply and octuply) excited states by perturbation methods, they obtain $E = -27.09$ Mev for the convergence limit. With the approximation degree $\lambda_{\frac{1}{2}}$ we obtain $B_{\frac{1}{2}} = 35.37$ Mev in the former and $B_{\frac{1}{2}} = 35.53$ Mev in the latter case, with the effect of the exchange forces included. We thus find their convergence limit to lie still somewhat too high.

From a comparison of the calculations on H^3 and He^4 the authors mentioned conclude that the adopted values of the force constants yield, with small modifications, results which are not decisively incompatible. Support can be given to this conclusion also if more recent values of the binding energies (MATTAUCH-FLÜGGE, 1942) are used. If the values

$$a = 2.25 \cdot 10^{-13} \text{ cm}, g = 0.174$$

are adopted we obtain with the same assumptions as above

$$\text{H}^3: B_1 = 35.63 \text{ Mev for } E = -8.38 \text{ Mev}$$

$$\text{He}^4: B_{\frac{1}{2}} = 35.88 \text{ Mev for } E = -28.19 \text{ Mev}$$

In the last value the Coulomb energy $E_c = 0.83$ Mev is included.

A factor essential for the compatibility of the results is the value of the range a . This fact is important if the results of Margenau and Tyrell are compared with those from the very accurate calculations by RARITA and PRESENT (1937) on H^3 and He^4 for the exponential function potential. These authors report that the binding energy of the α -particle proves to be too large with their choice of the force constants. It is naturally impossible

directly to compare the force constants of different potential functions. There is, however, another mode of procedure. A comparative study of different potential types has been made by HOISINGTON, SHARE and BREIT (1939) in order to explain the experimental data on the scattering of protons by protons. Even if we are not ready to accept the equal-force hypothesis, we may employ these calculations to establish a correspondence between the range of the exponential function potential and that of the error function potential. In the former case the value assumed by Rarita and Present is adopted in the scattering analysis and found to fit available data. In the case of the error function potential, however, the range $2.25 \cdot 10^{-13}$ cm, assumed by Margenau and collaborators, must be diminished to $1.91 \cdot 10^{-13}$ cm. It is then necessary to repeat the calculations on H^3 and He^4 for the latter value of the range. Assuming $g = 0$, we find the following values of the force constant:

$$\text{H}^3: B_{\frac{1}{2}} = 37.67 \text{ Mev,}$$

$$\text{He}^4: B_{\frac{1}{2}} = 36.21 \text{ Mev.}$$

The force constant in the scattering analysis has the value $A = 26.28$ Mev. By an adjustment of the value of g this value can be made compatible with either of the values of B for H^3 or He^4 . Bearing in mind that the spin eigenfunctions are approximately antisymmetrical in the scattering, we can determine g from the approximate formula

$$\frac{A}{1 - 2g} = \frac{B_{\frac{1}{2}}}{1 - g} \quad (14.1)$$

whence $g = 0.232$. Obviously the corresponding value of the ratio (14.1) for He^4 is too low. This implies that a law of force which yields a correct binding energy for the triton, and is concordant with the $p - p$ scattering data, gives too large a binding energy for the α -particle. On this point the conclusions of Rarita and Present agree with ours. Furthermore, since the force constants for H^3 and He^4 depend on the parameter g approximately in the same way, our result is relatively independent of the accurate value of the scattering constant A .

CHAPTER V.

The Yukawa Potential.

15. The Two-body Problem.

In the meson theories of nuclear forces an important rôle is played by the so-called Yukawa potential

$$V(\mathbf{r}) = -C \cdot \frac{e^{-\kappa r}}{\kappa r} \quad (15.1)$$

The range defined by the quantity $1/\kappa$ is a characteristic property of this potential, which differs from most other potentials used in nuclear theory in having a singularity for $r=0$. It has been applied to the two-body problem by several authors, but since the discovery of the electric quadripole moment of the deuteron it has lost some of its interest. For the discussion of the three- and four-body problems it is, however, very useful to have a detailed knowledge of the corresponding two-body problem, all the more as the latter problem is fairly manageable with our methods of calculation. We need only make a summary of the results.

The integral equation for a two-body problem with the potential energy (15.1) is written

$$(p_0^2 + p^2) \varphi(\mathbf{p}) = \lambda \cdot \frac{\kappa}{2\pi^2} \int \frac{1}{\kappa^2 + (\mathbf{p} - \mathbf{p}')^2} \varphi(\mathbf{p}') d\mathbf{p}' \quad (15.2)$$

where

$$p_0^2 = -ME \text{ and } \lambda = CM/\kappa^2.$$

The transformation of the potential energy (15.1) into momentum space occurs in quantum theory in various connexions. (See, for example, KRAMERS (1938), p. 164.)

With the simple initial function (for the ground state)

$$\varphi_0(\mathbf{p}) = \frac{\beta}{\pi} \cdot \frac{1}{(\beta^2 + p^2)^2} \quad (15.3)$$

we compute by means of calculus of residues the following quantities

$$T_0 = \frac{p_0^2 + \beta^2}{8\beta^3} \quad (15.4)$$

$$\Phi_0 = \frac{\kappa}{2} \cdot \frac{1}{(2\beta + \kappa)^2} \quad (15.5)$$

$$T_1 = \frac{\kappa^2}{4} \cdot \frac{1}{(\beta + \kappa)(\beta + \kappa + p_0)^2} \quad (15.6)$$

$$\Phi_1 = \frac{\kappa^3}{2} \cdot \frac{1}{(\beta + \kappa + p_0)^2 (\beta + \kappa - p_0)^2} \cdot \log \frac{(\beta + 2\kappa + p_0)^2}{(\kappa + 2p_0)(2\beta + 3\kappa)} \quad (15.7)$$

The iterated functions are

$$\varphi_1(\mathbf{p}) = \frac{\kappa}{2\pi} \cdot \frac{1}{p_0^2 + p^2} \cdot \frac{1}{(\beta + \kappa)^2 + p^2} \quad (15.8)$$

$$\varphi_2(\mathbf{p}) = \frac{\kappa^2}{2\pi} \cdot \frac{1}{(\beta + \kappa)^2 - p_0^2} \cdot \frac{1}{p_0^2 + p^2} \cdot \frac{1}{p} \cdot \operatorname{arctg} \frac{(\beta + \kappa - p_0)p}{(\kappa + p_0)(\beta + 2\kappa) + p^2} \quad (15.9)$$

Even T_2 can be calculated, the result being somewhat involved,

$$T_2 = \frac{\kappa^4}{2} \cdot \frac{1}{(\beta + \kappa + p_0)^2 (\beta + \kappa - p_0)^2} \left[B\left(\frac{\beta + 3\kappa + p_0}{\beta + 2\kappa + p_0}\right) + B\left(\frac{\beta + 3\kappa + p_0}{\kappa + 2p_0}\right) - B\left(\frac{2\beta + 4\kappa}{\beta + 2\kappa + p_0}\right) - B\left(\frac{2\kappa + 2p_0}{\kappa + 2p_0}\right) - \frac{1}{2} \left(\log \frac{\beta + 2\kappa + p_0}{\kappa + 2p_0} \right)^2 \right] \quad (15.10)$$

The function $B(x)$ is defined by the indefinite integral

$$B(x) = \int_1^x \frac{\log t}{t-1} dt \quad (15.11)$$

Some series expansions of $B(x)$ are given in Appendix II.

The minimum of $\lambda_0 = T_0 / \Phi_0$ is obtained for a value of β determined by the equation

$$p_0^2 = \beta^2 \cdot \frac{2\beta - \kappa}{2\beta + 3\kappa} \quad (15.12)$$

The results of the numerical calculations are shown in Table II.

TABLE II.

p_0/κ	λ_0	$\lambda_{\frac{1}{2}}$	λ_1	$\lambda_{\frac{3}{2}}$	λ_{LH}
0	2.00	1.687 ₆	1.6805	1.6799 ₄	1.6799
0.25	2.31	2.240	2.2346	2.2334 ₇	2.2331
0.5	2.81	2.773 ₅	2.7702	2.7695 ₁	2.7692
1	3.83	3.819	3.8171	3.8167 ₇	3.8166
1.5	4.85	4.848	4.8472	4.8470 ₂	4.8470
2	5.87	5.869	5.8685	5.8683 ₈	5.8683

In the last column we reproduce the values of λ computed by HULTHÉN (1942), who employed an accurate variational treatment. Table II confirms our earlier observation that the iteration is more effective for small values of p_0/κ , that is for short-range forces. For large values of p_0/κ the variational method (λ_0) suffices in itself.

The minimum of $\lambda_{\frac{1}{2}} = \Phi_0/T_1$ occurs for a value of β determined by

$$p_0 = (\beta + \kappa) \frac{2\beta - \kappa}{2\beta + 3\kappa} \quad (15.13)$$

For $p_0/\kappa = 0.25$ we obtain the set

$$\lambda_{\frac{1}{2}} = 2.233, \quad \lambda_1 = 2.2336, \quad \lambda_{\frac{3}{2}} = 2.2331.$$

For purposes of comparison, the minimum of λ_1 is computed for the same value of p_0/κ . The result is

$$\lambda_1 = 2.2335, \quad \lambda_{\frac{3}{2}} = 2.23308_8.$$

16. The Ground States of H^3 and He^4 .

By the formulation of the three- and four-body problems in momentum space for the Yukawa potential, we make the same assumptions concerning the spin-dependence as in the case of the error function potential. We replace all integrals of the form

$$\int \exp \left\{ -\frac{a^2}{4} (\mathbf{p} - \mathbf{p}')^2 \right\} \varphi(\cdot, \mathbf{p}', \cdot) d\mathbf{p}' \quad (16.1)$$

in the equations (10.2) and (12.1) by integrals of the form

$$\int \frac{1}{\kappa^2 + (\mathbf{p} - \mathbf{p}')^2} \varphi(\cdot, \mathbf{p}', \cdot) d\mathbf{p}' \quad (16.2)$$

Further we choose the same trial wave functions as in the case of the error function potential, Equations (10.7) and (12.3), whence the quantities T_0 are given by Equations (10.12) and (12.6), while a connexion between the quantities Φ_0 in the two cases »error» and »Yukawa» can be established by means of the equality

$$\frac{1}{\kappa^2 + (\mathbf{p} - \mathbf{p}')^2} = \int_0^\infty dt \exp \{-\kappa^2 t - (\mathbf{p} - \mathbf{p}')^2 t\} \quad (16.3)$$

If $\frac{a^2}{4}$ in $\Phi_0(\text{error})$ is replaced by t , we may multiply $\Phi_0(\text{error})$ by $\exp \{-\kappa^2 t\}$ and integrate with respect to t in order to obtain $\Phi_0(\text{Yukawa})$. In this way we find

$$\text{for } \text{H}^3: \quad \lambda_0 = \frac{2}{3} \cdot \frac{1}{1-g} \cdot \sqrt{\frac{\pi}{2}} \cdot \frac{\gamma p_0^2 + \frac{3}{4}\gamma}{1 - \gamma F(\gamma)} \quad (16.4)$$

$$\text{and for } \text{He}^4: \quad \lambda_0 = \frac{1}{2} \cdot \frac{1}{1-g} \cdot \sqrt{\frac{\pi}{2}} \cdot \frac{\gamma p_0^2 + \frac{3}{4}\gamma}{1 - \gamma F(\gamma)} \quad (16.5)$$

where $\gamma = \frac{1}{2} b\kappa$ and $F(\gamma)$ is the error integral.

The minimum of λ_0 occurs for a value of the parameter γ which is determined by the equation

$$\frac{1}{1 - \gamma F(\gamma)} = \gamma^2 + \frac{2}{1 + \mu\gamma^2} \quad (16.6)$$

where

$$\mu = \frac{2p_0^2}{3\kappa^2} \text{ for } \text{H}^3 \text{ and } \mu = \frac{4p_0^2}{9\kappa^2} \text{ for } \text{He}^4.$$

It is useful to have the corresponding formulae for the two-body problem (for $g = 0$).

$$\lambda_0 = \sqrt{\frac{\pi}{2}} \cdot \frac{\gamma p_0^2 + \frac{3}{4}\gamma}{\kappa^2 + 1 - \gamma F(\gamma)} \quad (16.7)$$

$$\mu = \frac{4p_0^2}{3\kappa^2}$$

Calculation of $\lambda_{\frac{1}{2}}$. For the two-body problem the quantity T_1 is expressed by the integral

$$T_1 = \frac{\kappa^2}{4\pi^4} \int \frac{d\mathbf{p}}{p_0^2 + p^2} \int \frac{d\mathbf{p}'}{\kappa^2 + (\mathbf{p} - \mathbf{p}')^2} \cdot \exp \left\{ -\frac{b^2}{4} p'^2 \right\} \int \frac{d\mathbf{p}''}{\kappa^2 + (\mathbf{p} - \mathbf{p}'')^2} \exp \left\{ -\frac{b^2}{4} p''^2 \right\} \quad (16.8)$$

which can be transformed into

$$T_1 = \frac{\kappa^2}{4} V \pi \int_0^\infty du \int_0^\infty ds \int_0^\infty dt \exp \{ -p_0^2 u - \kappa^2 (s+t) \} A_{ust}^{-\frac{3}{2}} \quad (16.9)$$

where

$$A_{ust} = u \left(s + \frac{b^2}{4} \right) \left(t + \frac{b^2}{4} \right) + st \frac{b^2}{2} + (s+t) \frac{b^4}{16} \quad (16.10)$$

After integration with respect to $s-t$ and introduction of new variables we have

$$T_1 = \frac{4\kappa^2}{b^3} V \sqrt{2\pi} \int_0^\infty du \int_0^\infty dx \exp \left\{ -\frac{b^2}{2} (p_0^2 u + \kappa^2 x) \right\} \cdot \frac{x}{x+1} \cdot \frac{1}{ux+u+x} \cdot \frac{1}{(2ux+u+x)^{\frac{1}{2}}} \quad (16.11)$$

We compute the integral with respect to x , which is a function of u , by series expansion. The integral with respect to u is then computed numerically by means of Simpson's formula. Thus we find

$$T_1 = \frac{4\kappa^2}{b^3} V \sqrt{2\pi} \int_0^\infty du \exp \left\{ -\frac{b^2 p_0^2}{2} u \right\} \cdot \frac{1}{u+1} \cdot \frac{1}{(2u+1)^{\frac{1}{2}}} \cdot K(u) \quad (16.12)$$

where

$$K(u) = (u+1) K_1(u) - u K_2(u) \quad (16.13)$$

$$K_1(u) = \int_0^\infty dx \exp \left\{ -\frac{b^2 \kappa^2}{2} x \right\} \cdot \frac{1}{x+1} \left(x + \frac{u}{2u+1} \right)^{-\frac{1}{2}} \quad (16.14)$$

$$K_2(u) = \int_0^\infty dx \exp \left\{ -\frac{b^2 \kappa^2}{2} x \right\} \cdot \left(x + \frac{u}{u+1} \right)^{-1} \left(x + \frac{u}{2u+1} \right)^{-\frac{1}{2}} \quad (16.15)$$

The latter integrals are computed for $0 < u \leq \infty$ by extending the integrations to the singularity $x = -\frac{u}{2u+1}$. The chief part of the integral is then

$$\begin{aligned}\overline{K}_1(u) &= \int_{-\frac{u}{2u+1}}^{\infty} dx \cdot \exp \left\{ -\frac{b^2 \kappa^2}{2} x \right\} \cdot \frac{1}{x+1} \cdot \left(x + \frac{u}{2u+1} \right)^{-\frac{1}{2}} = \\ &= 2 \int_0^{\infty} dz \cdot \exp \left\{ -\frac{b^2 \kappa^2}{2} \left(z^2 - \frac{u}{2u+1} \right) \right\} \left(z^2 + \frac{u+1}{2u+1} \right)^{-1} = \\ &= \sqrt{2\pi} \cdot \left(\frac{2u+1}{u+1} \right)^{\frac{1}{2}} \exp \left\{ \frac{b^2 \kappa^2}{2} \cdot \frac{u}{2u+1} \right\} F \left(b\kappa \sqrt{\frac{u+1}{2u+1}} \right) \quad (16.16)\end{aligned}$$

where $F(x)$ is the error integral. A similar result is obtained for $K_2(u)$. The overlapping part of the integral which must be subtracted from $\overline{K}_1(u)$ in order to obtain $K_1(u)$ is calculated by series expansion of the exponential function and integration term-by-term.

Accordingly we put

$$K_1(u) = \overline{K}_1(u) - \overline{\overline{K}}_1(u) \quad (16.17)$$

where

$$\overline{\overline{K}}_1(u) = \int_{-\frac{u}{2u+1}}^0 dx \cdot \exp \left\{ -\frac{b^2 \kappa^2}{2} x \right\} \frac{1}{x+1} \left(x + \frac{u}{2u+1} \right)^{-\frac{1}{2}} \quad (16.18)$$

Replacing x by $\frac{(u+1)z^2 - u}{2u+1}$ we find

$$\overline{\overline{K}}_1(u) = 2 \left(\frac{2u+1}{u+1} \right)^{\frac{1}{2}} \cdot \exp \left\{ \frac{b^2 \kappa^2}{2} \cdot \frac{u}{2u+1} \right\} \cdot J(u) \quad (16.19)$$

where

$$J(u) = \int_0^{\sqrt{\frac{u}{u+1}}} dz \cdot \exp \left\{ -\frac{b^2 \kappa^2}{2} \cdot \frac{u+1}{2u+1} \cdot z^2 \right\} \cdot \frac{1}{z^2 + 1} \quad (16.20)$$

If we expand the exponential function in the integrand and integrate term-by-term, we obtain

$$J(u) = \sum_{v=0}^{\infty} \frac{k_v}{v!} \left(-\frac{b^2 \kappa^2}{2} \cdot \frac{u+1}{2u+1} \right)^v \quad (16.21)$$

where the coefficients k_v are defined by

$$k_v = \int_0^{\sqrt{\frac{u}{u+1}}} dz \cdot \frac{z^{2v}}{z^2 + 1} \quad (16.22)$$

and are calculated by means of the recurrence formula

$$k_{v+1} = \frac{1}{2v+1} \left(\frac{u}{u+1} \right)^{v+\frac{1}{2}} - k_v \quad (16.23)$$

$$k_0 = \arctg \left(\frac{u}{u+1} \right)^{\frac{1}{2}}$$

We give the results of our calculations only for two values of p_0/κ .

p_0/κ	λ_0	$\lambda_{\frac{1}{2}}$	λ (correct)
0	2.71	1.737	1.680
0.25	2.95	2.323	2.233

The table shows that the error function is a poor approximation to the true eigenfunction for the Yukawa potential with a short range $1/\kappa$. The improvements achieved by the iteration amount, in the two cases, to 94 and 87 per cent of the error in λ_0 . The corresponding figures in Table II of Section 15 are 98 and 91 per cent.

The calculation of T_1 for the three- and four-body problems is in the main analogous to the corresponding calculations in Sections 10 and 12 for the error function potential. Among the nine integrals of which T_1 for the triton is composed, three are equal to

$$T_1(13|13) = \frac{\kappa^2}{4\pi^4} \int \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{p_0^2 + p_1^2 + p_2^2 + \mathbf{p}_1 \cdot \mathbf{p}_2} \int \frac{d\mathbf{p}'_1}{\kappa^2 + (\mathbf{p}_1 - \mathbf{p}'_1)^2}.$$

$$\begin{aligned}
& \cdot \exp \left\{ -\frac{b^2}{4} (p_1'^2 + p_2'^2 + \mathbf{p}_1' \cdot \mathbf{p}_2') \right\} \int \frac{d\mathbf{p}_1''}{\kappa^2 + (\mathbf{p}_1 - \mathbf{p}_1'')^2} \cdot \\
& \cdot \exp \left\{ -\frac{b^2}{4} (p_1''^2 + p_2'^2 + \mathbf{p}_1'' \cdot \mathbf{p}_2') \right\} = \\
& = \frac{2^7 \pi^2 \kappa^2}{3b^6 \sqrt{3}} \int_0^\infty du \int_0^\infty dx \exp \left\{ -\frac{b^2}{2} (p_0'^2 u + \kappa^2 x) \right\} \cdot \frac{x}{x+1} \cdot \\
& \cdot \frac{1}{(u+1)^{\frac{3}{2}}} \cdot \frac{1}{ux+u+x} \cdot \frac{1}{(2ux+u+x)^{\frac{1}{2}}} \quad (16.24)
\end{aligned}$$

and six equal to

$$\begin{aligned}
T_1(13|23) &= \frac{\kappa^2}{4\pi^4} \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{p_0'^2 + p_1'^2 + p_2'^2 + \mathbf{p}_1' \cdot \mathbf{p}_2'} \int \frac{d\mathbf{p}_1'}{\kappa^2 + (\mathbf{p}_1 - \mathbf{p}_1')^2} \cdot \\
& \cdot \exp \left\{ -\frac{b^2}{4} (p_1'^2 + p_2'^2 + \mathbf{p}_1' \cdot \mathbf{p}_2') \right\} \int \frac{d\mathbf{p}_2'}{\kappa^2 + (\mathbf{p}_2 - \mathbf{p}_2')^2} \cdot \\
& \cdot \exp \left\{ -\frac{b^2}{4} (p_1'^2 + p_2'^2 + \mathbf{p}_1' \cdot \mathbf{p}_2') \right\} = \\
& = \frac{2^7 \pi^2 \kappa^2}{3b^6 \sqrt{3}} \int_0^\infty du \int_0^\infty dx \exp \left\{ -\frac{b^2}{2} (p_0'^2 u + \kappa^2 x) \right\} \cdot \frac{1}{(x+1)(u+1) - \frac{1}{4}} \cdot \\
& \cdot \frac{1}{(x+1)(u+1) - \frac{3}{4}} \cdot \frac{1}{\left[(2x+1)(u+1)^2 - (x+1)(u+1) + \frac{3}{16} \right]^{\frac{1}{2}}} \quad (16.25)
\end{aligned}$$

The numerical computations of these integrals have been performed according to those methods which have been outlined above for the two-body problem.

For the four-body problem there are, correspondingly, three different integrals $T_1(14|14)$, $T_1(14|24)$ and $T_1(12|34)$.

$$\begin{aligned}
T_1(14|14) &= \frac{\kappa^2}{4\pi^4} \iiint \frac{d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_3}{p_0'^2 + p_1'^2 + p_2'^2 + p_3'^2 + \mathbf{p}_1' \cdot \mathbf{p}_2' + \mathbf{p}_1' \cdot \mathbf{p}_3' + \mathbf{p}_2' \cdot \mathbf{p}_3'} \cdot \\
& \cdot \int \frac{d\mathbf{p}_1'}{\kappa^2 + (\mathbf{p}_1 - \mathbf{p}_1')^2} \cdot \exp \left\{ -\frac{b^2}{4} (p_1'^2 + p_2'^2 + p_3'^2 + \mathbf{p}_1' \cdot \mathbf{p}_2' + \mathbf{p}_1' \cdot \mathbf{p}_3' + \mathbf{p}_2' \cdot \mathbf{p}_3') \right\} \cdot
\end{aligned}$$

$$\cdot \int \frac{d\mathbf{p}_1''}{\kappa^2 + (\mathbf{p}_1 - \mathbf{p}_1'')^2} \exp \left\{ -\frac{b^2}{4} (p_1''^2 + p_2^2 + p_3^2 + \mathbf{p}_1'' \cdot \mathbf{p}_2 + \right. \\ \left. + \mathbf{p}_1'' \cdot \mathbf{p}_3 + \mathbf{p}_2 \cdot \mathbf{p}_3) \right\} = \frac{2^7 \pi^{\frac{3}{2}} \kappa^2}{b^9} \int_0^\infty du \int_0^\infty dx \exp \left\{ -\frac{b^2}{2} (p_0^2 u + \kappa^2 x) \right\} \cdot \\ \cdot I_{ux}(14|14) \quad (16.26)$$

where

$$I_{ux}(14|14) = \frac{x}{x+1} \cdot \frac{1}{(u+1)^3} \cdot \frac{1}{ux+u+x} \cdot \frac{1}{(2ux+u+x)^{\frac{3}{2}}} \quad (16.27)$$

Analogous expressions are obtained for the other integrals:

$$I_{ux}(14|24) = \frac{x}{(x+1)(u+1) - \frac{1}{4}} \cdot \frac{1}{(x+1)(u+1) - \frac{3}{4}} \cdot \\ \cdot \frac{1}{\left[(2x+1)(u+1)^2 - (x+1)(u+1) + \frac{3}{16} \right]^{\frac{1}{2}}} \quad (16.28)$$

$$I_{ux}(12|34) = \frac{x}{(u+1)^{\frac{3}{2}}} \cdot \frac{1}{\left(ux+u+x+\frac{1}{2} \right)^2} \cdot \\ \cdot \frac{1}{\left[(2x+1)(u+1)^2 - (x+1)(u+1) + \frac{1}{4} \right]^{\frac{1}{2}}} \quad (16.29)$$

The similarity of all these expressions entails a great simplification of the numerical treatment.

17. Discussion of the Results.

In this section we shall summarize the results of the numerical work which has been carried out on the lines sketched in the last section. Regarding the two-body problem, we have seen already that the approximation $\lambda_{\frac{1}{2}}$ gives a value which is about four per cent too high. We must presume that matters are not so favourable in the case of the triton. Fortunately one very accurate calculation of the triton problem exists, namely that by BROWN (1939). To start with we check our methods by adopting the same values of the nuclear parameters as did Brown. The parameters are, in our notation,

$$1/\kappa = 1.835 \cdot 10^{-13} \text{ cm}, \quad g = 0.21, \quad E = -8.33 \text{ Mev.}$$

According to Equation (16.4) we find for the value of the force constant C (Equation 15.1)

$$C_0 = 37.21 \text{ Mev}$$

and by evaluation of the integrals (16.24) and (16.25)

$$C_{\frac{1}{2}} = 33.80 \text{ Mev}$$

This figure is to be compared with Brown's value

$$C = 32.06 \text{ Mev}$$

The discrepancy is somewhat more than five per cent. It is several times larger than the corresponding estimated difference for the error function potential.

Brown's work comprised also the calculation of the ground state of the deuteron with the same nuclear parameters. The compatibility of the results for the two- and three-particle problems may, however, justify Brown's choice of the nuclear parameters only in so far as the non-central forces in the deuteron can be neglected. (*Cf.* BROWN and PLESSET 1939.) Apart from this consideration it is useful to apply Brown's parameter values to the four-body problem. According to the approximate formula (16.5) we find

$$C_0 = 32.89 \text{ Mev}$$

which, though higher than Brown's value $C = 32.06$ for the triton, is apparently too low. The calculation of $C_{\frac{1}{2}}$ by means of the formulae (16.26—29) confirms this supposition. We obtain

$$C_{\frac{1}{2}} = 29.69 \text{ Mev.}$$

The convergence limit is probably in the vicinity of 28 Mev.

A certain significance must be attributed to this discrepancy, which cannot be removed by a reasonable change in the parameter g , since $C(\text{H}^3)$ and $C(\text{He}^4)$ depend on g in nearly the same way. Compatible results cannot be anticipated except with a much higher value of the range $1/\kappa$. The assumption of a wider range is, however, contradictory to the results obtained from the analysis of the proton-proton scattering experiments (HOISINGTON, SHARE and BREIT 1939). Though this analysis cannot, at least for the present, reveal the accurate shape of the potential curve for the

proton-proton force, it has given strong support to the Yukawa potential, with preference for the parameter values

$$\begin{aligned} 1/\kappa &= 1.185 \cdot 10^{-13} \text{ cm,} \\ C &= 45.76 \text{ Mev.} \end{aligned}$$

If we calculate C for H^3 and He^4 with this new value of $1/\kappa$ and put $g=0$, we find in the same way as above

$$\begin{aligned} C_0 &= 62.51 \text{ Mev, } C_{\frac{1}{2}} = 56.21 \text{ Mev for } \text{H}^3 \\ \text{and } C_0 &= 52.53 \text{ Mev for } \text{He}^4. \end{aligned}$$

The convergence limit of $C(\text{H}^3)$ can be estimated to be about 53.4 Mev. These figures show unambiguously that only one of the quantities $C(\text{H}^3)$ and $C(\text{He}^4)$ can be made compatible with the scattering constant $C(pp)$ by a suitable choice of the parameter g . Thus we may take (*cf.* Equation 14.1)

$$\frac{45.76}{1-2g} = \frac{53.4}{1-g} \quad (17.1)$$

whence $g=0.125$. Though this estimation is a very rough one, the discrepancy between $C(\text{H}^3)$ and $C(\text{He}^4)$ cannot be removed by any other choice of the parameter g .

The conclusions which can be drawn from these results agree exceedingly well with the analogous conclusions in Section 14 regarding the error function potential and with the results of RARITA and PRESENT (1937) for the exponential function potential. They may be summed up in the following manner. If the range of the nuclear forces is assumed on the basis of the analysis on the $p-p$ scattering data, the binding energies of H^3 and He^4 cannot be fitted by a single set of the remaining force parameters, comprehending spin exchange forces as well as other exchange forces, which are compatible with the equal-force hypothesis. It is satisfactory that this conclusion is independent of the special shape of the potential curve in the three somewhat different cases examined. It is also noteworthy that the inconsistency seems to be most pronounced for the Yukawa potential which, of the various potentials examined, has the shape which best fits the $p-p$ scattering data.

It may be mentioned that HYLLERAAS and RISBERG (1942) have reported an inconsistency for the Yukawa potential of the same

significance as in our case. It could not be expected, however, that their results would be sufficiently reliable, as they were founded on a very rough estimation of the binding energies of H^3 and He^4 .

Finally we may briefly discuss our results in the light of the meson theories, by way of which the Yukawa potential was first introduced into nuclear theory. The important relation between the mass M_m of a meson and the range $1/\kappa$ of the nuclear forces

$$M_m = \frac{\kappa \hbar}{c}$$

gives, for a value of κ derived from the $p-p$ scattering data, $M_m = 326 m$, where m is the mass of the electron. This value is nearly twice as large as the most probable value obtained from the experimental mass determinations. Brown's value, $1/\kappa = 1.835 \cdot 10^{-13}$ cm, corresponds to the mass $M_m = 211 m$, which accords much better with the experimental values. Neither of these values of κ , however, is compatible simultaneously with the binding energies of H^3 and He^4 , as we have seen above.

The inconsistency might possibly be removed by choosing a considerably higher value of $1/\kappa$ and a relatively high value of g . The latter condition is not, for example, fulfilled in the non-symmetrical scalar-pseudoscalar meson theory of HULTHÉN (1944).

It has been mentioned in the Introduction that a difference possibly exists between the $p-p$ force and the $p-n$ force, as they are determined from the scattering analysis. Since the ratio between the number of $p-n$ interactions and that of $p-p$ and $n-n$ interactions is the same for H^3 and He^4 , a small difference of the kind mentioned cannot change our conclusions.

In their analysis of the $p-p$ scattering data HOISINGTON, SHARE and BREIT emphasize that their results are rather sensitive to velocity-dependent forces. It is quite obvious that such forces may have some influence upon the binding energies too. It might be impossible, however, to draw any conclusions regarding this question if velocity-dependent forces were introduced only for *a priori* reasons.

Relativistic effects have been neglected throughout our discussion. On this point the reader is referred to the work of RARITA and PRESENT (1937).

Chapter VI.

The Hypothesis of Many-body Forces.

18. It has already been mentioned in Chapter III, Section 7, that the assumption of forces only between all pairs of particles in a nucleus might imply too great a simplification of the real state of the nucleus and that the intimate connexion between the nuclear force field and the motion of the nuclear constituents could be accounted for by the assumption of many-body forces. In this chapter we shall discuss such forces in order to find a way out of the dilemma which resulted from the preceding discussion. Many-body forces have been derived from different types of meson theories by PRIMAKOFF and HOLSTEIN (1939) and by JÁNOSSY (1939). It is by no means necessary to go into the details of these calculations. We only notice that the potential energy of two nucleons 1 and 2 is determined not only by their separation r_{12} and their spin directions in relation to the vector $\mathbf{r}_{12}$ but also by the positions and spin directions of other nucleons. If the equal-force hypothesis is maintained, we find for the total potential energy of the triton, for example, an expression

$$V = V(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23})$$

which is a symmetrical function with respect to the three vectors $\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}$. If the angle-dependence of the potential is not too pronounced, it can be concluded that an *attractive* three-body force should *increase* the binding energies of the triton and of the α -particle. Furthermore, the increase should be larger for the α -particle. We could verify these statements by calculations with the variational or the V—I method, but we do not think this to be necessary. Since the binding energy of the α -particle proves to be already too large for two-body forces, we are obliged to assume

repulsive three-body forces. If such forces of appropriate strength be introduced beside the ordinary two-body forces, it should be possible to eliminate the inconsistency between the binding energies of H^3 and He^4 , while preserving the values for the strength of the two-body forces obtained from the scattering experiments. At all events we must suppose that the latter forces produce the major part of the interaction energy. For the α -particle we cannot exclude four-body forces *a priori*, but a still smaller rôle must be attributed to these than to the three-body forces and it is not possible to surmise from the binding energies whether they are repulsive or attractive. We think that it is pointless for the present to introduce three-body forces of any special type in order to calculate more or less accurately the binding energies of H^3 and He^4 . Instead we ought to infer what consequences the hypothesis of many-body forces entails for the binding energies of heavier nuclei.

It is a well-known fact that neither ordinary forces nor spin exchange forces can explain the characteristic saturation property of the binding energies for nuclei heavier than the α -particle. The saturation has been ascribed to exchange forces of the Heisenberg or Majorana type or possibly to strong repulsive forces at small distances. A satisfactory proof of the existence or non-existence of these types of force has not been given hitherto. At first sight one might apprehend that the introduction of many-body forces would make the situation even worse than in the case of ordinary forces. The potential energy of a nucleus bound by attractive three-body forces should increase with the mass number A approximately as A^3 . Repulsive three-body forces with any noticeable influence upon the binding of the triton would be disastrous for the existence of a somewhat heavier nucleus. It is naturally always possible to endow the many-body forces with exchange properties of various kinds in order to secure the saturation of energy. In those cases when they are derived from a meson theory, the many-body forces are conditioned in most cases by the exchange properties of the meson field. If, however, we cease for the moment to consider the possible origin of the many-body forces, we must in any case require that their actions compensate each other in some way despite their unfavourable properties of forcible augmentation with the number of particles.

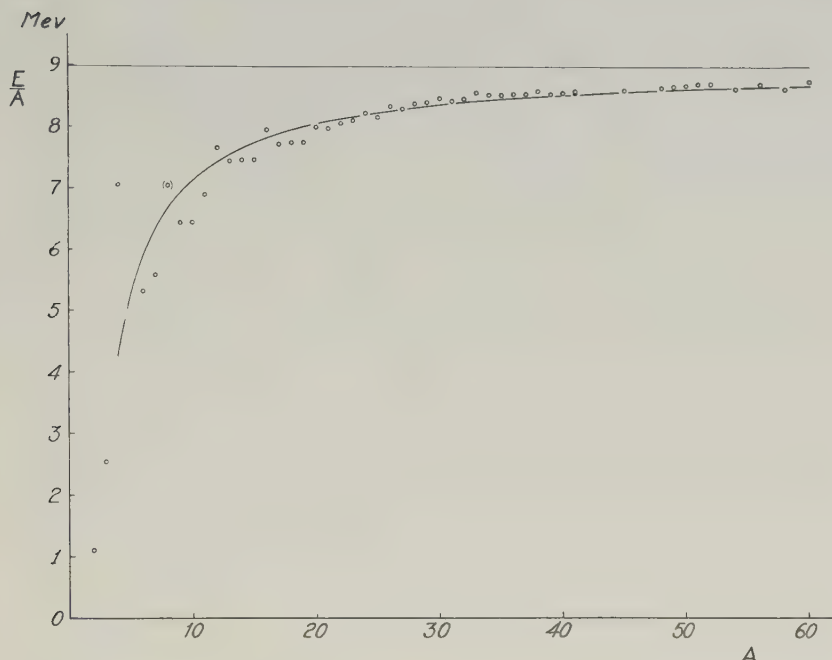


Figure 4. The binding energy per particle E/A as a function of the mass number A .

If the suggestion of repulsive three-body forces in the triton be taken as an indication of a simple model, we may assume that the main part of the binding energy of a nucleus with the mass number A , which is due to the action of ordinary two-body forces and proportional to $\binom{A}{2}$, is diminished by the action of repulsive three-body forces. It seems justifiable to put their contribution to the binding energy proportional to $-\gamma \binom{A}{3}$ where γ is a dimensionless quantity smaller than unity. For n -body forces we put, somewhat arbitrarily, a similar expression, namely $(-\gamma)^{n-2} \binom{A}{n}$. The quantity γ corresponds in some measure to the interaction parameter $g^2/\hbar c$ of the meson theories. The total binding energy is expressed by

$$E = \varepsilon \sum_{n=2}^A (-\gamma)^{n-2} \binom{A}{n} \quad (18.1)$$

where ε is a parameter with the dimension of an energy. For the binding energy per particle we find from (18.1)

$$\frac{E}{A} = \varepsilon \cdot \frac{(1-\gamma)^A - 1 + A\gamma}{A\gamma^2} \quad (18.2)$$

The term $(1-\gamma)^A$ can be neglected if γ is not too close to zero and A is not too small, whence

$$\frac{E}{A} = \frac{\varepsilon}{\gamma} \left(1 - \frac{1}{A\gamma} \right) \quad (18.3)$$

This expression shows two characteristic features of the binding energy as a function of the mass number: the increase of E/A with A for lighter nuclei and the constancy of E/A for heavier nuclei. In Figure 4 we have plotted the function (18.3) against A with the parameter values $\varepsilon = 4.35$ Mev, $\gamma = 0.484$, obtained by an adjustment to all known binding energies of stable nuclei up to $A = 60$. The curve shows a remarkable agreement with the real binding energies, but it must be remembered that our mode of consideration is very rough and that the action of the many-body forces can be inferred in many other ways.

APPENDIX I.

Calculation of the Matrix Elements $T_1(lmn|l'm'n')$.

The matrix elements $T_1(lmn|l'm'n')$ are defined by

$$T_1(lmn|l'm'n') = \int d\mathbf{p}'' \frac{1}{p_0^2 + p''^2} \int d\mathbf{p} \exp \left\{ -\frac{k}{4} (\mathbf{p} - \mathbf{p}'')^2 \right\} \cdot \\ \cdot \varphi_l(p_x) \varphi_m(p_y) \varphi_n(p_z) \int d\mathbf{p}' \exp \left\{ -\frac{k}{4} (\mathbf{p}' - \mathbf{p}'')^2 \right\} \varphi_{l'}(p'_x) \varphi_{m'}(p'_y) \varphi_{n'}(p'_z) \quad (\text{AI.1})$$

The oscillator wave functions $\varphi_l(p_x)$ etc. are defined by

$$\varphi_l(p_x) = N_l \exp \left\{ -\frac{1}{2} p_x^2 \right\} H_l(p_x) \quad (\text{AI.2})$$

where

$$N_l = (\sqrt{\pi} \cdot 2^l l!)^{-\frac{1}{2}}$$

is the normalization factor.

In order to calculate the integrals with respect to $\mathbf{p}, \mathbf{p}', \mathbf{p}''$, we introduce the Hermite polynomials by the integral representation

$$H_l(p_x) = \frac{l!}{2\pi i} \int_C \frac{dw}{w^{l+1}} \exp \{ 2p_x w - w^2 \} \quad (\text{AI.3})$$

where the contour integral C encloses the origin. As usual we put

$$\frac{1}{p_0^2 + p''^2} = \int_0^\infty ds \exp \{ -(p_0^2 + p''^2)s \} \quad (\text{AI.4})$$

The integrals with respect to $\mathbf{p}, \mathbf{p}', \mathbf{p}''$ are then computed in the Cartesian coordinates $p_x p_y \dots$. For this purpose we make use of the formula (for $n = 3$)

$$\int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_n \exp \left\{ -\sum_{jk} a_{jk} x_j x_k + 2 \sum_j b_j x_j \right\} = \\ = \pi^{\frac{n}{2}} A^{-\frac{1}{2}} \exp \left\{ A^{-1} \sum_{jk} b_j b_k A_{jk} \right\} \quad (\text{AI.5})$$

where

$$A = \det a_{jk}$$

and A_{jk} are the minors of (a_{jk}) .

The result can be written

$$T_1(lmn|l'm'n') = \frac{k}{(2+k)^3} \left[\frac{l! m! n! l'! m'! n'!}{2^{l+m+n+l'+m'+n'}} \right] \int_0^\infty ds \exp \{-p_0^2 s\} \cdot \\ \cdot \left(s + \frac{k}{2+k} \right)^{-\frac{3}{2}} \frac{1}{(2\pi i)^6} \int_{C'} \int_{C''} \frac{d\mathbf{w} d\mathbf{w}'}{w_x^{l'+1} w_y^{m'+1} w_z^{n'+1} (w'_x)^{l'+1} (w'_y)^{m'+1} (w'_z)^{n'+1}} \cdot \\ \cdot \exp \left\{ \frac{2-k}{2+k} (\mathbf{w}^2 + \mathbf{w}'^2) + \left(\frac{k}{2+k} \right)^2 (\mathbf{w} + \mathbf{w}')^2 \left(s + \frac{k}{2+k} \right)^{-1} \right\} \quad (\text{AI.6})$$

The integration variables w_x, w_y, w_z are understood as the components of a vector $\mathbf{w}$. By expanding the last exponential function into a series, we obtain from the term with index ν all matrix elements for which

$$l+m+n+l'+m'+n' = 2\nu$$

For all such elements we have the expression

$$\frac{T_1(lmn|l'm'n')_{2\nu}}{T_1(000|000)} = \frac{(l! m! n! l'! m'! n'!)^{\frac{1}{2}}}{2^\nu \nu!} \cdot \left(\frac{2-k}{2+k} \right)^\nu \cdot \\ \cdot \frac{1}{(2\pi i)^6} \int_{C'} \int_{C''} \frac{d\mathbf{w} d\mathbf{w}'}{w_x^{l'+1} \dots (w'_z)^{n'+1}} \sum_{\mu=0}^\nu \binom{\nu}{\mu} (\mathbf{w}^2 + \mathbf{w}'^2)^{\nu-\mu} (\mathbf{w} + \mathbf{w}')^{2\mu} g_\mu \quad (\text{AI.7})$$

where

$$g_\mu = \left(\frac{k}{2+k} \right)^{\nu-\mu} \frac{G_{\mu+\frac{3}{2}}(\xi)}{G_{\frac{3}{2}}(\xi)} \quad \text{and} \quad \xi = \frac{k p_0^2}{2+k}$$

The functions $G_n(\xi)$ are defined by

$$G_n(\xi) = \int_0^\infty dt e^{-t} \left(\frac{\xi}{\xi + t} \right)^n \quad (\text{AI.8})$$

and satisfy the recurrence formula

$$G_{n+1}(\xi) = \frac{\xi}{n} [1 - G_n(\xi)] \quad (\text{AI.9})$$

$$G_{\frac{1}{2}}(\xi) = \sqrt{2\xi} F(\sqrt{2\xi})$$

The function $G_{\frac{3}{2}}(\xi)$ has been used in Section 4.

It is convenient to reproduce an analogous expression for the matrix elements of Φ_0

$$\frac{\Phi_0(lmn|l'm'n')_{2\nu}}{\Phi_0(000|000)} = \frac{(l! m! n! l'! m'! n')^{\frac{1}{2}}}{2^\nu \cdot \nu!} \cdot \frac{1}{(1-k)^\nu} \cdot \frac{1}{(2\pi i)^6} \cdot \int_C \int_{C'} \frac{d\mathbf{w} d\mathbf{w}'}{w_x^{l+1} \dots (w'_z)^{n'+1}} \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} (\mathbf{w}^2 + \mathbf{w}'^2)^{\nu-\mu} (\mathbf{w} + \mathbf{w}')^{2\mu} \left(\frac{k}{1-k} \right)^\mu \quad (\text{AI.10})$$

The similarity of the expressions (AI.7) and (AI.10) with regard to the variables $\mathbf{w}, \mathbf{w}'$ is made use of in the evaluation of the contour integrals, which is carried out by calculus of residues.

APPENDIX II.

Calculation of the Function $B(x)$.

The function $B(x)$, defined by the integral

$$B(x) = \int_1^x \frac{\log t}{t-1} dt \quad (\text{AII.1})$$

can be calculated by means of the functional equations

$$B(x) + B\left(\frac{1}{x}\right) = \frac{1}{2}(\log x)^2 \quad (\text{AII.2})$$

$$B(x) + B(1-x) = \log x \log (1-x) - \frac{\pi^2}{6} \quad (\text{AII.3})$$

combined with the series expansion

$$B(x) = - \sum_{r=1}^{\infty} \frac{(1-x)^r}{r^2} \quad \text{valid for } |1-x| \leq 1 \quad (\text{AII.4})$$

Much more convenient for the numerical calculation is, however, the expansion

$$B(x) = \log x + \frac{1}{4}(\log x)^2 + \sum_{r=1}^{\infty} (-1)^{r-1} \frac{B_r}{(2r+1)!} (\log x)^{2r+1} \quad (\text{AII.5})$$

which is absolutely and uniformly convergent for $|\log x| < 2\pi$. B_r = Bernoullian numbers (WHITTAKER and WATSON, A Course of Modern Analysis, Cambridge 1935, p. 125).

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